The Weekly Rigor

No. 2 "A mathematician is a machine for turning coffee into theorems." July 5, 2014

Basic Logarithmic Properties
(Part 2)

Theorem 5 (Product Rule)**:** $\log_a(x \bullet y) = \log_a(x) + \log_a(y)$.

Preliminary Remark: In words: The log of a product equals the sum of the logs of the two factors.

Proof:

$$a^{\log_a(x) + \log_a(y)} = a^{\log_a(x)} \bullet a^{\log_a(y)} = x \bullet y,$$

by Theorem 4. Therefore,

$$\log_a(x \bullet y) = \log_a(x) + \log_a(y),$$

by Definition 1.

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Example: $\log_2(32) = \log_2(8 \bullet 4) = \log_2(8) + \log_2(4) = \log_2(2^3) + \log_2(2^2) = 3 + 2 = 5.$

Theorem 6 (Quotient Rule)**:** $\log_a(x/y) = \log_a(x) - \log_a(y)$.

Preliminary Remark: In words: The log of a quotient equals the log of the numerator minus the log of the denominator.

Proof:

$$a^{\log_a(x) - \log_a(y)} = a^{\log_a(x)}/a^{\log_a(y)} = x/y,$$

by Theorem 4. Therefore,

$$\log_a(x/y) = \log_a(x) - \log_a(y),$$

by Definition 1.

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Theorem 7 (Power Rule)**:** $\log_a(x^m) = m \cdot \log_a(x)$.

Preliminary Remark: In words: The log of a power equals the exponent times the log of the power's base.

Proof: $x^m = [a^{\log_a(x)}]^m$,

by Theorem 4.

$[a^{\log_a(x)}]^m = a^{\log_a(x) \cdot m} = a^{m \cdot \log_a(x)}$.

Hence, $a^{m \cdot \log_a(x)} = x^m$.

Therefore, $\log_a(x^m) = m \cdot \log_a(x)$,

by Definition 1.

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Theorem 8 (Change of Base Formula)**:** $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$.

Proof: $\log_b(x) = \log_b(a^{\log_a(x)})$,

by Theorem 4.

$\log_b(a^{\log_a(x)}) = \log_a(x) \cdot \log_b(a)$,

by Theorem 7. Hence,

$\log_a(x) \cdot \log_b(a) = \log_b(x)$.

Therefore,

$\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$.

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Example: $\log_2(x) = \log_e(x)/\log_e(2) = \log_{10}(x)/\log_{10}(2)$.