The Weekly Rigor

No. 3 "A mathematician is a machine for turning coffee into theorems." July 12, 2014

Some Consequences of the Basic Logarithmic Properties
(Part 1)

INTRODUCTION

In the following, the italicized lower-case Roman letters a, b, c shall stand for any real positive numbers other than 1. They shall be used for bases. The letter m shall stand for any real number. It shall be used for exponents. The letters x, y, z shall stand for any positive real numbers. They shall be used as terms in the arguments of the logarithmic functions. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception.

Theorem 1 (taking logs of equals)**:** If

$$x = y,$$

then

$$\log_a(x) = \log_a(y).$$

Preliminary Remark: In words: Equal quantities have equal logs (of the same base).

Proof: Suppose that

$$x = y.$$

Hence,

$$a^{\log_a(x)} = a^{\log_a(y)},$$

by Theorem 4 of *WR* no. 1. So,

$$\log_a(a^{\log_a(y)}) = \log_a(x),$$

by Definition 1 of *WR* no. 1. Thus,

$$\log_a(y) = \log_a(x),$$

by Theorem 4 of *WR* no. 1. Therefore,

$$\log_a(x) = \log_a(y).$$

■

Remark: This theorem is really an expression that logarithms are *functions*.

Theorem 2 (eliminating logs)**:** If

$$\log_a(x) = \log_a(y),$$

then

$$x = y.$$

Preliminary Remark: In words: Equal logs have equal arguments.

Proof: Suppose that

$$\log_a(x) = \log_a(y).$$

Hence,

$$a^{\log_a(y)} = x$$

and

$$a^{\log_a(x)} = y,$$

by Definition 1 of *WR* no. 1.

$$a^{\log_a(x)} = a^{\log_a(x)}.$$

So,

$$a^{\log_a(y)} = a^{\log_a(x)},$$

by substitution. Therefore,

$$x = y.$$

■

Remark: This theorem is really an expression that logarithms are *one-to-one* functions.

Theorem 3 (Change of Base for Exponentials)**:**

$$a^m = b^{m \bullet \log_b(a)}.$$

Proof:

$$a^m = b^{\log_b(a^m)},$$

by Theorem 4 of *WR* no. 1.

$$b^{\log_b(a^m)} = b^{m \bullet \log_b(a)},$$

by Theorem 7 of *WR* no. 2. Therefore,

$$a^m = b^{m \bullet \log_b(a)}.$$

■

Example: $2^m = e^{m \bullet \ln(2)} = 10^{m \bullet \log_{10}(2)}$. (Note: $\ln(x) = \log_e(x)$.)

Problem: Express $5^x \bullet \log_9(x)$ in base e.

Solution:

$$5^x = e^{x \bullet \ln(5)},$$

by Theorem 3.

$$\log_9(x) = \ln(x)/\ln(9),$$

by Theorem 8 of *WR* no. 2. Therefore,

$$5^x \bullet \log_9(x) = e^{x \bullet \ln(5)} \bullet \ln(x)/\ln(9).$$