The Weekly Rigor

No. 4 | "A mathematician is a machine for turning coffee into theorems." | July 19, 2014

Some Consequences of the Basic Logarithmic Properties
(Part 2)

Theorem 4: $\log_a(x/y) + \log_a(y/x) = 0.$

Proof: $\log_a(x/y) = \log_a(x) - \log_a(y)$

and

$$\log_a(y/x) = \log_a(y) - \log_a(x),$$

by Theorem 6 of *WR* no. 2. Therefore,

$$\log_a(x/y) + \log_a(y/x) = [\log_a(x) - \log_a(y)] + [\log_a(y) - \log_a(x)]$$
$$= \log_a(x) - \log_a(y) + \log_a(y) - \log_a(x) = 0.$$

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Theorem 5: $\log_a(x^{\log_a(x)}) = [\log_a(x)]^2.$

Proof: $\log_a(x^{\log_a(x)}) = \log_a(x) \cdot \log_a(x),$

by Theorem 7 of *WR* no. 1.

$$\log_a(x) \cdot \log_a(x) = [\log_a(x)]^2.$$

Therefore,

$$\log_a(x^{\log_a(x)}) = [\log_a(x)]^2.$$

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Theorem 6: $\log_a(x^{\log_a(x^{\log_a(x)})}) = [\log_a(x)]^3.$

Proof: $\log_a(x^{\log_a(x^{\log_a(x)})}) = \log_a(x^{[\log_a(x)]^2}),$

by Theorem 5.

$$\log_a(x^{[\log_a(x)]^2}) = [\log_a(x)]^2 \cdot \log_a(x),$$

by Theorem 7 of *WR* no. 1.

$$[\log_a(x)]^2 \cdot \log_a(x) = [\log_a(x)]^3.$$

Therefore,

$$\log_a(x^{\log_a(x^{\log_a(x)})}) = [\log_a(x)]^3.$$

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Theorem 7:

$$\log_a(x + y) = \log_a(x) + \log_a(1 + y/x).$$

Proof:

$$\log_a(x + y) = \log_a(x + x \bullet [y/x]) = \log_a(x \bullet [1 + y/x]).$$

But,

$$\log_a(x \bullet [1 + y/x]) = \log_a(x) + \log_a(1 + y/x),$$

by Theorem 5 of *WR* no. 2. Therefore,

$$\log_a(x + y) = \log_a(x) + \log_a(1 + y/x).$$

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Theorem 8:

$$\log_a(x + y) = \log_a(x) + \log_a(1 + a^{\log_a(y) - \log_a(x)}).$$

Proof:

$$\log_a(x + y) = \log_a(x) + \log_a(1 + y/x),$$

by Theorem 7.

$$a^{\log_a(y)} = y$$

and

$$a^{\log_a(x)} = x,$$

by Theorem 4 of *WR* no. 1. Hence,

$$\log_a(x) + \log_a(1 + y/x) = \log_a(x) + \log_a(1 + a^{\log_a(y)}/a^{\log_a(x)})$$
$$= \log_a(x) + \log_a(1 + a^{\log_a(y) - \log_a(x)}).$$

Therefore,

$$\log_a(x + y) = \log_a(x) + \log_a(1 + a^{\log_a(y) - \log_a(x)}).$$

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Theorem 9: For $x > y$,

$$\log_a(x - y) = \log_a(x) + \log_a(1 - y/x).$$

Preliminary Remark: The purpose of the constraint $x > y$ is to prevent $x - y \leq 0$ and $1 - y/x \leq 0$ in the arguments of the logarithmic functions, which, per Definition 1 of *WR* no. 1, are forbidden.

Proof: For $x > y$,

$$\log_a(x - y) = \log_a(x + [-y]).$$

But,

$$\log_a(x + [-y]) = \log_a(x) + \log_a(1 + [-y]/x),$$

by Theorem 7.

$$\log_a(x) + \log_a(1 + [-y]/x) = \log_a(x) + \log_a(1 - y/x).$$

Therefore,

$$\log_a(x - y) = \log_a(x) + \log_a(1 - y/x).$$

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