The Weekly Rigor

No. 5 | "A mathematician is a machine for turning coffee into theorems." | July 26, 2014

Some Consequences of the Basic Logarithmic Properties

(Part 3)

Theorem 10: For $x > y$,

$$\log_a(x - y) = \log_a(x) + \log_a(1 - a^{\log_a(y) - \log_a(x)}).$$

Proof: For $x > y$,

$$\log_a(x - y) = \log_a(x) + \log_a(1 - y/x),$$

by Theorem 9.

$$a^{\log_a(y)} = y$$

and

$$a^{\log_a(x)} = x,$$

by Theorem 4 of *WR* no. 1. Hence,

$$\log_a(x) + \log_a(1 - y/x) = \log_a(x) + \log_a(1 - a^{\log_a(y)}/a^{\log_a(x)})$$
$$= \log_a(x) + \log_a(1 - a^{\log_a(y) - \log_a(x)}).$$

Therefore,

$$\log_a(x - y) = \log_a(x) + \log_a(1 - a^{\log_a(y) - \log_a(x)}).$$

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Theorem 11:

$$\log_a(x) \cdot \log_b(a) = \log_b(x).$$

Proof: By Theorem 8 of *WR* no. 2.

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Theorem 12:

$$a^{\log_b(c)} = c^{\log_b(a)}.$$

Proof:

$$a^{\log_b(c)} = c^{\log_b(c) \cdot \log_c(a)},$$

by Theorem 3.

$$c^{\log_b(c) \cdot \log_c(a)} = c^{\log_c(a) \cdot \log_b(c)}.$$

But,

$$c^{\log_c(a) \cdot \log_b(c)} = c^{\log_b(a)},$$

by Theorem 11. Therefore,

$$a^{\log_b(c)} = c^{\log_b(a)}.$$

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Problem: Evaluate $10^{\log_{100}(9)}$.

Solution:

$$10^{\log_{100}(9)} = 9^{\log_{100}(10)},$$

by Theorem 12.

$$9^{\log_{100}(10)} = 9^{\log_{100}(100^{1/2})} = 9^{1/2},$$

by Theorem 3 of *WR* no. 1.

$$9^{1/2} = 3.$$

Therefore,

$$10^{\log_{100}(9)} = 3.$$

Theorem 13: $\log_a(b) = 1/\log_b(a)$.

Proof:

$$\log_a(b) = \log_b(b)/\log_b(a),$$

by Theorem 8 of *WR* no. 2.

$$\log_b(b) = 1,$$

by Theorem 1 of *WR* no. 1. Therefore,

$$\log_a(b) = 1/\log_b(a).$$

■

Theorem 14: $\log_a(b) \bullet \log_b(a) = 1$.

Proof: By Theorem 13.

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Theorem 15: $\log_a(x) \bullet \log_b(a) \bullet \log_c(b) = \log_c(x)$.

Proof: $\log_a(x) \bullet \log_b(a) \bullet \log_c(b) = [\log_a(x) \bullet \log_b(a)] \bullet \log_c(b) = \log_b(x) \bullet \log_c(b) = \log_c(x)$, by Theorem 11.

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