The Weekly Rigor

No. 6 "A mathematician is a machine for turning coffee into theorems." August 2, 2014

Some Consequences of the Basic Logarithmic Properties
(Part 4)

Theorem 16: $\log_a(x) \bullet \log_b(y) \bullet \log_c(z) = \log_b(x) \bullet \log_c(y) \bullet \log_a(z)$.

Proof: $\log_a(x) \bullet \log_b(y) \bullet \log_c(z) = [\log_b(x)/\log_b(a)] \bullet [\log_c(y)/\log_c(b)] \bullet [\log_a(z)/\log_a(c)]$,
by Theorem 8 of *WR* no. 2.

$$[\log_b(x)/\log_b(a)] \bullet [\log_c(y)/\log_c(b)] \bullet [\log_a(z)/\log_a(c)]$$
$$= [\log_b(x) \bullet \log_c(y) \bullet \log_a(z)]/[\log_b(a) \bullet \log_c(b) \bullet \log_a(c)]$$
$$= [\log_b(x) \bullet \log_c(y) \bullet \log_a(z)]/\log_a(a),$$

by Theorem 15.

$$[\log_b(x) \bullet \log_c(y) \bullet \log_a(z)]/\log_a(a) = [\log_b(x) \bullet \log_c(y) \bullet \log_a(z)]/1$$
$$= \log_b(x) \bullet \log_c(y) \bullet \log_a(z),$$

by Theorem 1 of *WR* no. 1. Therefore,

$$\log_a(x) \bullet \log_b(y) \bullet \log_c(z) = \log_b(x) \bullet \log_c(y) \bullet \log_a(z).$$

■

Theorem 17: For $x \neq 1$,

$$\log_a(x)/\log_{a \bullet b}(x) = 1 + \log_a(b).$$

Preliminary Remark: The purpose of the extra constraint $x \neq 1$ is to prevent $\log_{a \bullet b}(x) = 0$, which would render "$\log_a(x)/\log_{a \bullet b}(x)$" undefined.

Proof: For $x \neq 1$,

$$\log_a(x)/\log_{a \bullet b}(x) = \log_a(x) \bullet \log_x(a \bullet b),$$

by Theorem 13.

$$\log_a(x) \bullet \log_x(a \bullet b) = \log_a(x) \bullet [\log_x(a) + \log_x(b)],$$

by Theorem 5 of *WR* no. 2.

$$\log_a(x) \bullet [\log_x(a) + \log_x(b)] = \log_a(x) \bullet \log_x(a) + \log_a(x) \bullet \log_x(b) = 1 + \log_a(x) \bullet \log_x(b),$$

by Theorem 14.

$$1 + \log_a(x) \bullet \log_x(b) = 1 + \log_x(b) \bullet \log_a(x) = 1 + \log_a(b),$$

by Theorem 11. Therefore,

$$\log_a(x)/\log_{a \bullet b}(x) = 1 + \log_a(b).$$

■

Theorem 18: For $m \neq 0$,

$$\log_{a^m}(x) = \log_a(x)/m.$$

Proof: For $m \neq 0$,

$$\log_{a^m}(x) = \log_a(x)/\log_a(a^m),$$

by Theorem 8 of *WR* no. 2.

$$\log_a(x)/\log_a(a^m) = \log_a(x)/m,$$

by Theorem 3 of *WR* no. 1. Therefore,

$$\log_{a^m}(x) = \log_a(x)/m.$$

■

Theorem 19: For $m \neq 0$,

$$\log_{a^m}(x^m) = \log_a(x).$$

Preliminary Remark: "Base and argument are like powers."

Proof: For $m \neq 0$,

$$\log_{a^m}(x^m) = m \bullet \log_{a^m}(x),$$

by Theorem 7 of *WR* no. 2.

$$m \bullet \log_{a^m}(x) = m \bullet [\log_a(x)/m],$$

by Theorem 18.

$$m \bullet [\log_a(x)/m] = \log_a(x).$$

Therefore,

$$\log_{a^m}(x^m) = \log_a(x).$$

■

Example: $\log_{1000}(729) = \log_{10^3}(9^3) = \log_{10}(9)$