The Weekly Rigor

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Counterexamples to Some Common Logarithmic Errors

(Part 1)

Certain errors are commonly committed by the new student. The goal of this article is to convince the reader that certain operations involving logarithms are invalid and undependable. The method I use involves constructing counterexamples that are simple to grasp, employing the theorems proved in earlier articles along with basic claims such as

$$0 \neq 1.$$

As an example of this method applied to algebra, we could provide a counterexample to the erroneous claim that for all real numbers

$$(x + y)^2 = x^2 + y^2.$$

For if we set $x = y = 1$, we get the unhappy result

$$(1 + 1)^2 = 2^2 = 4 \neq 2 = 1 + 1 = 1^2 + 1^2.$$

In the following, the italicized lower-case Roman letter a shall stand for any real positive number other than 1. It shall be used for bases. The letter m shall stand for any real number. It shall be used for exponents. The letters x and y shall stand for any positive real numbers. They shall be used as terms in the arguments of the logarithmic functions. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception.

Counterexample 1: For some a, x, and y,

$$\log_a(x \bullet y) \neq \log_a(x) \bullet \log_a(y).$$

Proof: Suppose by contradiction that

$$\log_a(x \bullet y) = \log_a(x) \bullet \log_a(y).$$

Hence, with $a = x = 2$ and $y = 1$,

$$\log_2(2 \bullet 1) = \log_2(2) = \log_2(2) \bullet \log_2(1),$$

by substitution. But

$$\log_2(2) = 1 \text{ and } \log_2(1) = 0,$$

by Theorems 1 and 2 of *WR* no. 1. So,

$$1 = 1 \bullet 0 = 0.$$

Contradiction.

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Counterexample 2: For some a, x, and y,

$$\log_a(x + y) \neq \log_a(x) \cdot \log_a(y).$$

Proof: Suppose by contradiction that

$$\log_a(x + y) = \log_a(x) \cdot \log_a(y).$$

Hence, with $a = 2$ and $x = y = 1$,

$$\log_2(1 + 1) = \log_2(2) = \log_2(1) \cdot \log_2(1),$$

by substitution. But

$$\log_2(2) = 1 \text{ and } \log_2(1) = 0,$$

by Theorems 1 and 2 of *WR* no. 1. So,

$$1 = 0 \cdot 0 = 0.$$

Contradiction.

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Counterexample 3: For some a, x, and y,

$$\log_a(x + y) \neq \log_a(x) + \log_a(y).$$

Proof: Suppose by contradiction that

$$\log_a(x + y) = \log_a(x) + \log_a(y).$$

Hence, with $a = 2$ and $x = y = 1$,

$$\log_2(1 + 1) = \log_2(2) = \log_2(1) + \log_2(1),$$

by substitution. But

$$\log_2(2) = 1 \text{ and } \log_2(1) = 0,$$

by Theorems 1 and 2 of *WR* no. 1. So,

$$1 = 0 + 0 = 0.$$

Contradiction.

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Counterexample 4: For some a, x, and y,

$$\log_a(x) \cdot \log_a(y) \neq \log_a(x) + \log_a(y).$$

Proof: Suppose by contradiction that

$$\log_a(x) \cdot \log_a(y) = \log_a(x) + \log_a(y).$$

Hence, with $a = x = y = 2$,

$$\log_2(2) \cdot \log_2(2) = \log_2(2) + \log_2(2),$$

by substitution. But

$$\log_2(2) = 1,$$

by Theorem 1 of *WR* no. 1. So,

$$1 \cdot 1 = 1 = 1 + 1 = 2.$$

Contradiction.

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