The Weekly Rigor

No. 10 "A mathematician is a machine for turning coffee into theorems." August 30, 2014

Some Elementary Divisibility Properties
(Part 1)

INTRODUCTION

In the following, italicized lower-case Roman letters shall stand for any integers. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception. The following facts about integers will be used without special mention: If a is an integer, then so are $-a$ and $|a|$; and if a and b are integers, then so are $a + b$, $a - b$, and ab.

Definition 1:
$$a \mid b$$
if and only if $a \neq 0$ and there exists an integer c such that
$$ac = b.$$

For "$a \mid b$," b is said to be divisible by a, b is a multiple of a, a divides b, or a goes into b.

If $a \neq 0$ and b is not divisible by a, then we write
$$a \nmid b.$$

Examples: $2 \mid 6$, $4 \nmid 6$, $3 \nmid 4$, $2 \mid -4$, $5 \mid 0$, $1 \mid 3$, $3 \mid 3$.

Theorem 1: $a \mid b$ if and only if b/a is an integer.

Proof: Suppose that $a \mid b$. Hence, there exists an integer c such that $ac = b$, by Definition 1. So, $c = b/a$. Therefore, b/a is an integer.

Suppose that b/a is an integer. Hence, there exists an integer c such that $c = b/a$. So, $ac = b$. Therefore, $a \mid b$, by Definition 1.

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Theorem 2: $a \nmid b$ if and only if b/a is not an integer.

Proof: By Theorem 1.

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Theorem 3: $a \mid a$.

Proof: $a \bullet 1 = a$

for any a, including $a \neq 0$. Hence, there exists an integer c such that $ac = a$. Therefore,

$a \mid a$,

by Definition 1.

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Theorem 4: $1 \mid a$.

Proof: $1 \bullet a = a$,

and $1 \neq 0$. Hence, there exists an integer c such that $1 \bullet c = a$. Therefore,

$1 \mid a$,

by Definition 1.

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Theorem 5: $a \mid 1$ if and only if either $a = 1$ or $a = -1$.

Proof: Suppose that $a \mid 1$. Hence, $1 = ac$ for some c, by Definition 1. The only two possibilities for a and c are either $a = 1 = c$ or $a = -1 = c$. Therefore, either $a = 1$ or $a = -1$.

Suppose that either $a = 1$ or $a = -1$. Hence, either $a \bullet 1 = 1$ or $a \bullet (-1) = 1$. In either case, there exists an integer c such that $ac = 1$. Therefore,

$a \mid 1$,

by Definition 1.

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Theorem 6: $a \mid 0$.

Proof: $a \bullet 0 = 0$

for any a, including $a \neq 0$. Hence, there exists an integer c such that $ac = 0$. Therefore,

$a \mid 0$,

by Definition 1.

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Theorem 7: If both $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof: Suppose that both $a \mid b$ and $b \mid c$. Hence, $ad = b$ and $be = c$ for some d and e, by Definition 1. So, $(ad)e = c$, by substitution. Thus, $a(de) = c$. Therefore,

$a \mid c$,

by Definition 1.

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