The Weekly Rigor

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Some Elementary Divisibility Properties
(Part 2)

Theorem 8: If both $a \mid b$ and $a \nmid c$, then $b \nmid c$.

Proof: Suppose that both $a \mid b$ and $a \nmid c$. Hence, not both $a \mid b$ and $b \mid c$, by Theorem 7. So, either $a \nmid b$ or $b \nmid c$, by Definition 1. Therefore,

$$b \nmid c,$$

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Theorem 9: If both $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for any x and y.

Proof: Suppose that both $a \mid b$ and $a \mid c$. Hence, $ad = b$ and $ae = c$ for some d and e, by Definition 1. So, $adx = bx$ and $aey = cy$. Thus $a(dx + ey) = adx + aey = bx + cy$. Therefore,

$$a \mid (bx + cy),$$

by Definition 1.

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Theorem 10: If $a \mid b$, then $a \mid bx$ for any x.

Proof: Suppose that $a \mid b$. $a \mid 0$, by Theorem 6. Hence, $a \mid (bx + 0 \bullet y)$ for any x and y, by Theorem 9. Therefore,

$$a \mid bx.$$

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Theorem 11: If $a \nmid bc$, then both $a \nmid b$ and $a \nmid c$.

Proof: Suppose that either $a \mid b$ or $a \mid c$. Hence, in either case, $a \mid bc$, by Theorem 10. Therefore, if $a \nmid bc$, then neither $a \mid b$ nor $a \mid c$.

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Theorem 12: If $a \mid b$, then $ac \mid b$ for some c.

Proof: Suppose that $a \mid b$. Hence, $b = ac$ for some c, by Definition 1. So, $b = ac \bullet 1$. Thus, there exists an integer k such that $b = ack$. Therefore,

$$ac \mid b,$$

for some c, by Definition 1.

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Theorem 13: If $a \mid b$, then $ac \mid bx$ for some c and any x.

Proof: Suppose that $a \mid b$. Hence, $a \mid bx$ for any x, by Theorem 10. Therefore,

$$ac \mid bx,$$

for some c and any x, by Theorem 12.

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Theorem 14: If $a \nmid bc$, then both $a \nmid b$ and $a \nmid c$.

Proof: Suppose that either $a \mid b$ or $a \mid c$. WLOG, let $a \mid b$. Hence, $a \mid bc$, by Theorem 10. Therefore, If $a \nmid bc$, then neither $a \mid b$ nor $a \mid c$.

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Theorem 15: If both $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof: Suppose that both $a \mid b$ and $a \mid c$. Hence, $a \mid (b \bullet 1 + c \bullet 1)$, by Theorem 9. Therefore,

$$a \mid (b + c).$$

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Theorem 16: If both $a \mid b$ and $a \mid c$, then $a \mid (b - c)$ and $a \mid (c - b)$.

Proof: Suppose that both $a \mid b$ and $a \mid c$. Hence, $a \mid (b \bullet 1 + c \bullet [-1])$ and $a \mid (b \bullet [-1] + c \bullet 1)$ by Theorem 9. So, $a \mid (b - c)$ and $a \mid (-b + c)$. Therefore,

$$a \mid (b - c) \text{ and } a \mid (c - b).$$

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Theorem 17: If both $a \mid b$ and $a \mid (b + c)$, then $a \mid c$.

Proof: Suppose that both $a \mid b$ and $a \mid (b + c)$. Hence, $b = ak$ and $b + c = al$ for some k and l, by Definition 1. So, $ak + c = al$, by substitution. Thus, $c = al - ak = a(l - k)$. Therefore,

$$a \mid c,$$

by Definition 1.

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