The Weekly Rigor

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Some Elementary Divisibility Properties
(Part 4)

Theorem 28: If both $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.

Proof: Suppose that both $a \mid b$ and $b \mid a$. Hence, $ac = b$ and $bd = a$ for some c and d, by Definition 1. So, $(ac)d = a(cd) = a$, by substitution. Thus, $cd = 1$. Hence, either $c = 1 = d$ or $c = -1 = d$. Therefore, either $a = b$ or $a = -b$, by substitution.

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Theorem 29: If $a \mid b$, then $|a| \mid |b|$.

Proof: Suppose that $a \mid b$. Hence, $ac = b$ for some c, by Definition 1. So, $|ac| = |a|\bullet|c| = |b|$. Therefore,

$$|a| \mid |b|,$$

by Definition 1.

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Theorem 30: If both $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$.

Proof: Suppose that both $a \mid b$ and $b \neq 0$. Hence, $|a| \mid |b|$, by Theorem 29. So, $|a|c = |b|$ for some c, by Definition 1. Since $b \neq 0$, $|b| > 0$. Thus, $c > 0$. Therefore,

$$|a| \leq |b|.$$

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Theorem 31: If both a and b are positive integers and $a \mid b$, then $1 \leq a \leq b$.

Proof: Suppose that both a and b are positive integers and $a \mid b$. Hence, $b \neq 0$. So, $|a| \leq |b|$, by Theorem 30. Furthermore, $|a| = a$, $|b| = b$, and $1 \leq a$. Therefore,

$$1 \leq a \leq b,$$

by substitution.

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Theorem 32: If both a and b are positive integers and $a > b$, then $a \nmid b$.

Proof: Suppose that both a and b are positive integers and $a > b$. Suppose by contradiction that $a \mid b$. Hence, $1 < a \leq b$, by Theorem 31. So, $a \not> b$. Contradiction. Therefore,

$$a \nmid b.$$

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Theorem 33: If $a, b > 1$ and $a \mid b$, then $a \nmid (b + 1)$.

Proof: Suppose that $a, b > 1$ and $a \mid b$. Suppose by contradiction that $a \mid (b + 1)$. Hence, $a \mid 1$, by Theorem 17. So, $a \leq 1$, by Theorem 31. Thus, $a \not> 1$. Contradiction. Therefore,

$$a \nmid (b + 1).$$

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Theorem 34: If $a, b > 1$, $a \mid b$, and $a \mid (b + 2)$, then $a = 2$.

Proof: Suppose that $a, b > 1$, $a \mid b$, and $a \mid (b + 2)$. Hence, $a \mid 2$, by Theorem 17. So, $1 < a \leq 2$, by Theorem 31. Therefore, $a = 2$.

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Theorem 35: If $a, b > 1$, $a \mid b$, and $a \mid (b + 3)$, then $a = 3$.

Proof: Suppose that $a, b > 1$, $a \mid b$, and $a \mid (b + 3)$. Hence, $a \mid 3$, by Theorem 17. So, $1 < a \leq 3$, by Theorem 31. Thus, either $a = 2$ or $a = 3$. Suppose that $a = 2$. Hence, $2 \mid b$ and $2 \mid (b + 3)$, by substitution. $2 \mid 2$, by Theorem 3. So, $2 \mid (b + 2)$, by Theorem 15. Thus, $2 \mid ([b + 3] - [b + 2])$, viz., $2 \mid 1$, by Theorem 16. But since $2 > 1$, $2 \nmid 1$, by Theorem 32. Hence, $a \neq 2$. Therefore, $a = 3$.

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Theorem 36: If $a \mid b$ and $a \nmid c$, then $a \nmid (b + c)$.

Proof: Suppose that $a \mid b$ and $a \nmid c$. Hence, not both $a \mid b$ and $a \mid (b + c)$, by Theorem 17. So, either $a \nmid b$ or $a \nmid (b + c)$, by Definition 1. Therefore,

$$a \nmid (b + c).$$

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