The Weekly Rigor

No. 14 “A mathematician is a machine for turning coffee into theorems.” September 27, 2014

Even and Odd Integers: Basic Properties of Their Sums and Products

(Part 1)

INTRODUCTION

In the following, italicized lower-case Roman letters shall stand for any integers. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception. The following facts about integers will be used without special mention: If a and b are integers, then so are $a + b$ and ab.

Definition 1: An integer n is said to be *even* if $2 \mid n$.

Definition 2: An integer n is said to be *odd* if $2 \nmid n$.

Theorem 1: An integer n is either even or odd, but not both.

Proof: Either $2 \mid n$ or $2 \nmid n$. Hence, n is either even or odd, by Definitions 1 and 2. Suppose by contradiction that n is both even and odd. Hence, $2 \mid n$ and $2 \nmid n$. Contradiction. So, n is not both even and odd.

■

Theorem 2: An integer n is even if and only if there exists an integer k such that $n = 2k$.

Proof: Suppose that n is even. Hence, $2 \mid n$, by Definition 1. So, $n = 2k$ for some integer k, by Definition 1 of *WR* no. 10.

Suppose that there exists an integer k such that $n = 2k$. Hence, $2 \mid n$, by Definition 1 of *WR* no. 10. So, n is even, by Definition 1.

■

Theorem 3: The set of even integers is the set $E = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

Proof: $-6 = 2(-3)$, $-4 = 2(-2)$, $-2 = 2(-1)$, etc. Hence, the members of E are the even integers, by Theorem 2.

■

Theorem 4: If n is even, then $n + 1$ is odd.

Proof: Suppose that n is even. Hence, $2 \mid n$, by Definition 1. Suppose by contradiction that $n + 1$ is not odd. Hence, $n + 1$ is even, by Theorem 1. So, $2 \mid (n + 1)$, by Definition 1. But $2 \nmid 1$, by Theorem 31 of *WR* no. 13. Thus, $2 \nmid n$, by Theorem 17 of *WR* no. 11. Contradiction. Therefore, $n + 1$ is an odd integer.

■

Theorem 5: The set of odd integers is the set $O = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$.

Proof: Each member of O is one unit greater than an even integer. Hence, each member of O is an odd integer, by Theorem 4.

■

Theorem 6: Each member of the set $O = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$ can be expressed in the form $2k + 1$ for some integer k.

Proof: Each member of O is one unit greater than an even integer. Hence, each member of O can be expressed in the form $2k + 1$, by Theorem 2.

■

Theorem 7: An integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.

Proof: Suppose that n is an odd integer. Hence, there exists an integer k such that $n = 2k + 1$, by Theorems 5 and 6.

Suppose that n is an integer such that $n = 2k +1$ for some integer k. Hence, $n - 1 = 2k$. So, $2 \mid (n - 1)$, by Definition 1 of *WR* no. 10. But $2 \nmid 1$, by Theorem 31 of *WR* no. 13. Thus, $2 \nmid n$, by Theorem 21 of *WR* no. 12. Hence, n is odd, by Definition 2.

■

Theorem 8: $2 \nmid n$ if and only if $2 \mid (n + 1)$.

Proof: Suppose that $2 \nmid n$. Hence, n is odd, by Definition 2. So, $n = 2k + 1$ for some k, by Theorem 7. Thus, $n + 1 = 2k + 2 = 2(k + 1)$. Therefore, $2 \mid (n + 1)$, by Definition 1 of *WR* no. 10.

Suppose that $2 \mid (n + 1)$. Suppose by contradiction that $2 \mid n$. Hence, $2 \mid ([n + 1] + n)$, viz., $2 \mid (2n + 1)$, by Theorem 15 of *WR* no. 11. So, $2n + 1$ is even, by Definition 1, and $2n + 1$ is odd, by Theorem 7. But $2n + 1$ cannot be both even and odd, by Theorem 1. Contradiction. Therefore, $2 \nmid n$.

■