The Weekly Rigor

No. 15 "A mathematician is a machine for turning coffee into theorems." October 4, 2014

Even and Odd Integers: Basic Properties of Their Sums and Products
(Part 2)

Theorem 9: If both a and b are even, then $a + b$ is also even.

Proof: Suppose that both a and b are even. Hence, both $2 \mid a$ and $2 \mid b$, by Definition 1. So, $2 \mid (a + b)$, by Theorem 15 of *WR* no. 11. Therefore, $a + b$ is even, by Definition 1.

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Theorem 10: If a is even and b is odd, then $a + b$ is odd.

Proof: Suppose that a is even and b is odd. Hence, $2 \mid a$ and $2 \nmid b$, by Definitions 1 and 2. So, $2 \nmid (a + b)$, by Theorem 36 of *WR* no. 13. Therefore, $a + b$ is odd, by Definition 2.

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Theorem 11: If both a and b are odd, then $a + b$ is even.

Proof: Suppose that both a and b are odd. Hence, $2 \nmid a$ and $2 \nmid b$, by Definition 2. So, $2 \mid (a + 1)$ and $2 \mid (b + 1)$, by Theorem 8. Thus, $2 \mid ([a + 1] + [b + 1])$, viz., $2 \mid ([a + b] + 2)$, by Theorem 15 of *WR* no. 11. $2 \mid 2$, by Theorem 3 of *WR* no. 10. Hence, $2 \mid (a + b)$, by Theorem 17 of *WR* no. 11. Therefore, $a + b$ is even, by Definition 1.

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Remark: Theorems 9 to 11 can be summarized as follows. Given any even integer n, for any two integers that add up to n, either both are even or both are odd. Given any odd integer n, for any two integers that add up to n, one is even, the other is odd.

Theorem 12: If both a and b are even, then ab is even.

Proof: Suppose that both a and b are even. Hence, $2 \mid a$, by Definition 1. So, $2 \mid ab$, by Theorem 10 of *WR* no. 11. Therefore, ab is even, by Definition 1.

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Theorem 13: If a is even and b is odd, then ab is even.

Proof: Suppose that a is even and b is odd. Hence, $2 \mid a$, by Definition 1. So, $2 \mid ab$, by Theorem 10 of *WR* no. 11. Therefore, ab is even, by Definition 1.

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Remark: By an exactly similar argument, ab is even if a is odd and b is even.

Theorem 14: Both a and b are odd if and only if ab is odd.

Proof: Suppose that both a and b are odd. Hence, $a = 2k + 1$ and $b = 2l + 1$ for some k and l, by Theorem 7. So, $ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$. Therefore, ab is odd, by Theorem 7.

Suppose that ab is odd. Hence, ab is not even, by Theorem 1. So, not both a and b are even, by Theorem 12. Furthermore, it is not the case that a is even and b is odd, by Theorem 13. (And similarly it is not the case that a is odd and b is even.) Thus, neither a nor b is even. Therefore, both a and b are odd.

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Theorem 15: Either a or b is even if and only if ab is even.

Proof: By Theorems 14 and 1.

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Theorem 16: If ab is odd, then $a + b$ is even.

Proof: Suppose that ab is odd. Hence, both a and b are odd, by Theorem 14. Therefore, $a + b$ is even, by Theorem 11.

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The results of Theorems 9-16 can be summarized in a table, such as the following:

a	b	$a + b$	ab
EVEN	EVEN	EVEN	EVEN
EVEN	ODD	ODD	EVEN
ODD	EVEN	ODD	EVEN
ODD	ODD	EVEN	ODD