The Weekly Rigor

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Even and Odd Integers: Some Consequences of the Basic Properties
(Part 1)

INTRODUCTION

In the following, italicized lower-case Roman letters shall stand for any integers. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception. The following fact about integers will be used without special mention: If a and b are integers, then so are $a + b$ and ab.

Theorem 1: For any odd a, a^n is odd for positive integer n.

Proof: By induction on positive integers n. Let a be odd.

I. $a^1 = a$ is odd.

II. Suppose for positive integer k that a^k is odd. Hence, $a^{k+1} = a^k \bullet a^1$ is the product of two odd integers, by the induction hypothesis. Consequently, $a^{k+1} = a^k \bullet a^1$ is odd, by Theorem 14 of *WR* no. 15.

Therefore, the theorem holds, by mathematical induction.

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Theorem 2: For any even a, a^n is even for positive integer n.

Proof: By induction on positive integers n. Let a be even.

I. $a^1 = a$ is even.

II. Suppose for positive integer k that a^k is even. Hence, $a^{k+1} = a^k \bullet a^1$ is the product of two even integers, by the induction hypothesis. Consequently, $a^{k+1} = a^k \bullet a^1$ is even, by Theorem 15 of *WR* no. 15.

Therefore, the theorem holds, by mathematical induction.

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Theorem 3: For any odd a^n, where n is a positive integer, a is odd.

Proof: $a^n = a \bullet a^{n-1}$. Hence, both a and a^{n-1} are odd, by Theorem 14 of *WR* no. 15. So, a is odd.

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Theorem 4: For any positive integer n, a is odd if and only if a^n is odd.

Proof: Suppose a is odd. Hence, a^n is odd, by Theorem 1. Suppose a^n is odd. Hence, a is odd, by Theorem 3.

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Theorem 5: For any positive integer n, a is even if and only if a^n is even.

Proof: By Theorem 4 (above) and Theorem 1 of *WR* no. 14.

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Remark: The effect of Theorems 1-5 is that the positive integer degree of an integer is an irrelevant issue in determining whether that integer is even or odd. Hence, for example, if we wish to know whether the integer $a^{152}x^{97}$ is even or odd, we can direct our attention to the simpler expression ax and consult a table of the sort presented at the end of *WR* no. 15, viz.,

a	x	$a + x$	ax
EVEN	EVEN	EVEN	EVEN
EVEN	ODD	ODD	EVEN
ODD	EVEN	ODD	EVEN
ODD	ODD	EVEN	ODD

Examples: $4x^2 - 6x - 3$ is odd for any integer x, since if x is even, then so are $4x^2$ and $4x^2 - 6x$. Hence, $4x^2 - 6x - 3$ is odd. On the other hand, if x is odd, then $4x^2$, $6x$, and $4x^2 - 6x$ are even. So, $4x^2 - 6x - 3$ is odd.

$3x^5 + 7$ is even for any odd integer x, since if x is odd, then so is $3x^5$. Hence, $3x^5 + 7$ is even. On the other hand, $3x^5 + 7$ is odd for any even integer x, since if x is even, then so is $3x^5$. Hence, $3x^5 + 7$ is odd.

The product of two consecutive integers is always even, since exactly one of the two will be even. Hence, $x^2 + x + a$ will be odd if a is odd, since $x^2 + x + a = x(x + 1) + a$ is the sum of an even and an odd integer. Similarly, $x^2 + x + a$ will be even if a is even.

$a^n + a$ is even for any a and any positive integer n, since if a is even, then so are a^n and $a^n + a$. If a is odd, then so is a^n. Hence, $a^n + a$ is even.