The Weekly Rigor

No. 18 "A mathematician is a machine for turning coffee into theorems." October 25, 2014

Even and Odd Integers: Some Consequences of the Basic Properties
(Part 3)

The last table in *WR* no. 17 suggests some generalizations to expressions of any number of terms, each term being a product of two integers.

Definition 1: For integers $a_1, a_2, a_3, \ldots, a_n, x_1, x_2, x_3, \ldots, x_n$,

$$D_n = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n.$$

Theorem 6: If each term of D_n is even, then D_n is even.

Proof: By induction on positive integers n.

I. Suppose that each term of D_1 is even. Hence, since D_1 just consists of one term, D_1 is even.

II. Suppose for positive integer k that if each term of D_k is even, then D_k is even. Suppose that each term of D_{k+1} is even. $D_{k+1} = D_k + a_{k+1}x_{k+1}$. Hence, D_k is even, by the induction hypothesis. So, D_{k+1} is even, by Theorem 9 of *WR* no. 15. Consequently, if each term of D_{k+1} is even, then D_{k+1} is even.

Therefore, the theorem holds, by mathematical induction.

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Theorem 7: If D_n is odd, then at least one term of D_n is has two odd factors.

Proof: Suppose that D_n is odd. Hence, at least one term of D_n is odd, by Theorem 6. So, both the integer factors of that term are odd, by Theorem 14 of *WR* no. 15.

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Theorem 8: If each term of D_m is odd and m is even, then D_m is even.

Proof: By induction on positive integers n such that $m = 2n$.

I. Suppose that each term of D_2 is odd. Hence, since D_2 just consists of two terms, D_2 is even, by Theorem 11 of *WR* no. 15.

II. Suppose for positive integer k that if each term of D_m is odd and $m = 2k$ (viz., m is even), then D_m is even. Suppose that each term of D_{m+2} is odd and $m + 2 = 2k + 2 = 2(k + 1)$ (viz., $m + 2$ is even). $D_{m+2} = D_m + a_{m+1}x_{m+1} + a_{m+2}x_{m+2}$. Hence, D_m is even, by the induction hypothesis. But $a_{m+1}x_{m+1} + a_{m+2}x_{m+2}$ is even, by Theorem 11 of *WR* no. 15. So, D_{m+2} is even, by Theorem 9 of *WR* no. 15. Consequently, if each term of D_{m+2} is odd and $m + 2 = 2(k + 1)$, then D_{m+2} is even.

Therefore, the theorem holds, by mathematical induction.

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Theorem 9: If each term of D_m is odd and m is odd, then D_m is odd.

Proof: By induction on positive integers n such that $m = 2n + 1$.

I. Suppose that each term of D_1 is odd. Hence, since D_1 just consists of one term, D_1 is odd.

II. Suppose for positive integer k that if each term of D_m is odd and $m = 2k + 1$ (viz., m is odd), then D_m is odd. Suppose that each term of D_{m+2} is odd and $m + 2 = 2k + 3 = 2(k + 1) + 1$ (viz., $m + 2$ is odd). $D_{m+2} = D_m + a_{m+1}x_{m+1} + a_{m+2}x_{m+2}$. Hence, D_m is odd, by the induction hypothesis. But $a_{m+1}x_{m+1} + a_{m+2}x_{m+2}$ is even, by Theorem 11 of *WR* no. 15. So, D_{m+2} is odd, by Theorem 10 of *WR* no. 15. Consequently, if each term of D_{m+2} is odd and $m + 2 = 2(k + 1) +1$, then D_{m+2} is odd.

Therefore, the theorem holds, by mathematical induction.

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Theorem 10: If D_n has an equal number of even and odd terms and $n = 2k$ for some even k, then D_n is even.

Proof: Suppose that D_n has an equal number of even and odd terms and $n = 2k$ for some even k. Hence, there are k even terms and k odd terms of D_n, where k is even. So, the sum of the even terms is even, by Theorem 6. Furthermore, the sum of the odd terms is even, by Theorem 8. Thus, D_n is even, by Theorem 9 of *WR* no. 15.

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Theorem 11: If D_n has an equal number of even and odd terms and $n = 2k$ for some odd k, then D_n is odd.

Proof: Suppose that D_n has an equal number of even and odd terms and $n = 2k$ for some odd k. Hence, there are k even terms and k odd terms of D_n, where k is odd. So, the sum of the even terms is even, by Theorem 6. Furthermore, the sum of the odd terms is odd, by Theorem 9. Thus, D_n is odd, by Theorem 10 of *WR* no. 15.

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