The Weekly Rigor

No. 19 “A mathematician is a machine for turning coffee into theorems.” November 1, 2014

Even and Odd Integers: Some Consequences of the Basic Properties
(Part 4)

Theorem 12: a and b are either both even or both odd if and only if $(a + b)^2$ is even.

Proof: Suppose that a and b are either both even or both odd.

Case 1: Suppose that a and b are both even. Hence, $a + b$ is even, by Theorem 9 of *WR* no. 15. So, $(a + b)^2$ is even, by Theorem 2.

Case 2: Suppose that a and b are both odd. Hence, $a + b$ is even, by Theorem 11 of *WR* no. 15. So, $(a + b)^2$ is even, by Theorem 2.

In either case, $(a + b)^2$ is even.

Suppose that $(a + b)^2$ is even. Hence, $a + b$ is even, by Theorem 5. So, $a + b$ is not odd, by Theorem 1 of *WR* no. 14. Thus, a and b are either both even or both odd, by Theorem 10 of *WR* no. 15.

■

Theorem 13: If a, b, and c are all odd, then the roots of $ax^2 + bx + c = 0$ are not rational.

Proof: Suppose that $ax^2 + bx + c = 0$ $(a \neq 0)$ has a rational root. Hence, there exist integers m and n such that $x = m/n$ and $a(m/n)^2 + b(m/n) + c = 0$. So, $am^2 + bmn + cn^2 = 0$.

Suppose by contradiction that a, b, and c are all odd. Either both m and n are even or at least one of m, n is odd.

Case 1: Suppose that both m and n are even. Hence, am^2, bmn, and cn^2 are all even. So, $am^2 + bmn + cn^2$ is even (and $a \neq 0$). Thus, $2 \mid (am^2 + bmn + cn^2)$, by Definition 1 of *WR* no. 14. Hence, $2 \mid 0$, by substitution. But $2 \mid 0$, by Theorem 32 of *WR* no. 13. Contradiction.

Case 2: Suppose that at least one of m, n is odd. WLOG, let m by odd. Hence, since a is odd, am^2 is odd. So, since 0 is even, $bmn + cn^2$ is odd. Thus, either bmn is even and cn^2 is odd, or bmn is odd and cn^2 is even.

Case 2a: Suppose that bmn is even and cn^2 is odd. Hence, since both b and m are odd, n is even. So, since c is odd, cn^2 is even. Contradiction.

Case 2b: Suppose that bmn is odd and cn^2 is even. Hence, since bmn is odd, n is odd. So, since c is odd, cn^2 is odd. Contradiction.

In all cases, a contradiction results. Consequently, at least one of a, b, and c is not odd.

Therefore, by contraposition, if a, b, and c are all odd, then the roots of $ax^2 + bx + c = 0$ are not rational.

■

Theorem 14: For any primitive Pythagorean triple a, b, c such that $a^2 + b^2 = c^2$, exactly one of a, b is even and c is odd.

Preliminary Remark: A primitive Pythagorean triple is a Pythagorean triple where the three integers are coprime, i.e., the only common positive integer factor of the three numbers is 1.

Proof: Suppose that a, b, c is a primitive Pythagorean triple such that $a^2 + b^2 = c^2$. If a and b were both even, then c would also be even, by Theorem 9 of *WR* no. 15. But then a, b, and c would not be coprime, since they would share a common factor of 2. Hence, a and b cannot both be even.

Suppose by contradiction that both a and b are odd. Hence, c is even, by Theorem 11 of *WR* no. 15. So, $a = 2k + 1$, $b = 2l + 1$, and $c = 2m$, for some integers k, l, and m, by Theorems 2 and 7 of *WR* no. 14. Thus, $(2k + 1)^2 + (2l + 1)^2 = (2m)^2$, by substitution. Hence, $4k^2 + 4k + 1 +$ $+ 4l^2 + 4l + 1 = 4m^2$. So, $2k^2 + 2k + 2l^2 + 2l + 1 = 2m^2$. Thus, $2(k^2 + k + l^2 + l) + 1 = 2m^2$. Hence, the same number is both odd and even, by Theorems 2 and 7 of *WR* no. 14. Contradiction. Therefore, exactly one of a, b is even, and hence c is odd.

■

Theorem 15: $a - b$ is even if and only if $a + b$ is even.

Proof: Suppose that $a - b$ is even. Hence, $a - b = 2n$ for some n, by Theorem 2 of *WR* no. 14. So, $a + b = (2n + b) + b = 2(n + b)$. Thus, $a + b$ is even, by Theorem 2 of *WR* no. 14.

Suppose that $a + b$ is even. Hence, $a + b = 2m$ for some m, by Theorem 2 of *WR* no. 14. Suppose by contradiction that $a - b$ is odd. Hence, $a - b = 2n + 1$ for some n, by Theorem 7 of *WR* no. 14. So, $(2n + 1 + b) + b = 2m$. Thus, $2(n + b) + 1 = 2m$. Hence, the same number is both odd and even, by Theorems 2 and 7 of *WR* no. 14. Contradiction. Therefore, $a - b$ is even.

■

Theorem 16: If a is odd, then $a^2 = 8q + 1$ for some q.

Proof: Suppose that a is odd. Hence, $a = 2k + 1$ for some k, by Theorem 7 of *WR* no. 14. So, $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$. Hence, since $k(k + 1)$ is even, $k(k + 1) = 2q$ for some q, by Theorem 2 of *WR* no. 14. Therefore, $a^2 = 4(2q) + 1 = 8q + 1$, by substitution.

■

Theorem 17: If a and b are odd, then $a^2 - b^2 = 8q$ for some q.

Proof: Suppose that a and b are odd. Hence, $a = 2k + 1$ and $b = 2l + 1$ for some k and l by Theorem 7 of *WR* no. 14. So, $a^2 - b^2 = (2k + 1)^2 - (2l + 1)^2 = 4k^2 + 4k + 1 - (4l^2 + 4l + 1) =$ $= 4k^2 + 4k + 1 - 4l^2 - 4l - 1 = 4(k^2 - l^2 + k - l) = 4([k - l][k + l] + [k - l]) = 4(k - l)(k + l + 1)$. Exactly one of $(k - l)$ and $(k + l + 1)$ is even, by Theorem 15. Thus, either $a^2 - b^2 =$ $= 4(2n)(k + l + 1) = 8n(k + l + 1)$ or $a^2 - b^2 = 4(k - l)(2n) = 8n(k - l)$ for some n, by Theorem 2 of *WR* no. 14. Therefore, $a^2 - b^2 = 8q$ for some q.

■