The Weekly Rigor

No. 22 "A mathematician is a machine for turning coffee into theorems." November 22, 2014

51 Problems in Calculating Limits Using L'Hôpital's Rule with Solutions

(Part 3)

9. $$\lim_{t\to 0^+}\frac{\ln(t)}{\csc(t)} \overset{LH}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\csc(t)\cot(t)} = \lim_{t\to 0^+} -\frac{1}{t}\cdot\frac{1}{\csc(t)\cot(t)} = \lim_{t\to 0^+} -\frac{1}{t}\cdot\frac{1}{\frac{1}{\sin(t)}\cdot\frac{\cos(t)}{\sin(t)}} =$$
$$= \lim_{t\to 0^+} -\frac{1}{t}\cdot\frac{\sin^2(t)}{\cos(t)} = \lim_{t\to 0^+} -\frac{\sin(t)\tan(t)}{t} \overset{LH}{=} \lim_{t\to 0^+} -\frac{\cos(t)\tan(t)+\sin(t)\sec^2(t)}{1} =$$
$$= -\frac{\cos(0)\tan(0)+\sin(0)\sec^2(0)}{1} = -\frac{(1)(0)+(0)(1)}{1} = -\frac{0+0}{1} = 0.$$

10. $$\lim_{x\to 0}\frac{\cos(x)+2x-1}{3x} \overset{LH}{=} \lim_{x\to 0}\frac{-\sin(x)+2}{3} = \frac{-\sin(0)+2}{3} = \frac{-0+2}{3} = \frac{2}{3}.$$

11. $$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos(2x)} \overset{LH}{=} \lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin(2x)} \overset{LH}{=} \lim_{x\to 0}\frac{e^x+e^{-x}}{4\cos(2x)} = \frac{e^0+e^{-0}}{4\cos(0)} = \frac{1+1}{4(1)} = \frac{2}{4} = \frac{1}{2}.$$

12. $$\lim_{x\to\frac{\pi}{2}^-}\frac{4\tan(x)}{1+\sec(x)} \overset{LH}{=} \lim_{x\to\frac{\pi}{2}^-}\frac{4\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x\to\frac{\pi}{2}^-}\frac{4\sec(x)}{\tan(x)} = \lim_{x\to\frac{\pi}{2}^-}\frac{\left(\frac{4}{\cos(x)}\right)}{\left(\frac{\sin(x)}{\cos(x)}\right)} =$$
$$= \lim_{x\to\frac{\pi}{2}^-}\frac{4}{\sin(x)} = \frac{4}{\sin\left(\frac{\pi}{2}\right)} = \frac{4}{1} = 4.$$

13. $$\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty}\frac{\ln(x)}{x^{\frac{1}{2}}} \overset{LH}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2x^{\frac{1}{2}}}\right)} = \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to\infty}\frac{1}{x}\cdot\frac{2\sqrt{x}}{1} =$$
$$= \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

14. $$\lim_{x\to\infty}\frac{\ln(x)}{\ln(x+1)} \overset{LH}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+1}\right)} = \lim_{x\to\infty}\frac{x+1}{x} \overset{LH}{=} \lim_{x\to\infty}\frac{1}{1} = 1.$$

15. $$\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)-x+x^2}\overset{LH}{\cong}\lim_{x\to 0}\frac{\sin(x)}{\cos(x)-1+2x}\overset{LH}{\cong}\lim_{x\to 0}\frac{\cos(x)}{-\sin(x)+2}=\frac{\cos(0)}{-\sin(0)+2}=\frac{1}{-0\ +2}=$$
$$=\frac{1}{2}.$$

16. $$\lim_{x\to 0}\frac{\sin(x)-2x}{x}\overset{LH}{\cong}\lim_{x\to 0}\frac{\cos(x)-2}{1}=\frac{\cos(0)-2}{1}=\frac{1-2}{1}=-1.$$

17. $$\lim_{x\to\infty}\frac{\ln(x)}{x^{10}}\overset{LH}{\cong}\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{10x^9}=\lim_{x\to\infty}\frac{1}{10x^{10}}=0.$$

18. $$\lim_{x\to 0}\frac{10^x-2^x}{6x}\overset{LH}{\cong}\lim_{x\to 0}\frac{\ln(10)\,10^x-\ln(2)\,2^x}{6}=\frac{\ln(10)\,10^0-\ln(2)\,2^0}{6}=\frac{\ln(10)-\ln(2)}{6}=$$
$$=\frac{\ln\left(\frac{10}{2}\right)}{6}=\frac{\ln(5)}{6}.$$

19. $$\lim_{x\to 0}\frac{x^2}{\ln(\cos(x))}\overset{LH}{\cong}\lim_{x\to 0}\frac{2x}{\left(\frac{-\sin(x)}{\cos(x)}\right)}=\lim_{x\to 0}\frac{-2x\cos(x)}{\sin(x)}\overset{LH}{\cong}\lim_{x\to 0}\frac{-2\cos(x)+2x\sin(x)}{\cos(x)}=$$
$$=\frac{-2\cos(0)+2(0)\sin(0)}{\cos(0)}=\frac{-2(1)+2(0)(0)}{1}=\frac{-2}{1}=-2.$$

20. $$\lim_{x\to 0}\frac{x\cdot 2^x}{2^x-1}\overset{LH}{\cong}\lim_{x\to 0}\frac{2^x+x\ln(2)2^x}{\ln(2)\,2^x}=\lim_{x\to 0}\frac{2^x(1+x\ln(2))}{\ln(2)\,2^x}=\lim_{x\to 0}\frac{1+x\ln(2)}{\ln(2)}=\frac{1+(0)\ln(2)}{\ln(2)}=$$
$$=\frac{1}{\ln(2)}.$$

21. $$\lim_{x\to 0^+}\frac{\ln(x^2+2x)}{\ln(x)}\overset{LH}{\cong}\lim_{x\to 0^+}\frac{\left(\frac{2x+2}{x^2+2x}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to 0^+}\frac{2x+2}{x^2+2x}\cdot\frac{x}{1}=\lim_{x\to 0^+}\frac{2(x+1)x}{(x+2)x}=$$
$$=\lim_{x\to 0^+}\frac{2(x+1)}{(x+2)}=\frac{2(0+1)}{0+2}=\frac{2}{2}=1.$$

22. $$\lim_{x\to\infty}\frac{e^x+x^2}{e^x-x}\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x+2x}{e^x-1}\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x+2}{e^x}\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1.$$