The Weekly Rigor

No. 24 "A mathematician is a machine for turning coffee into theorems." December 6, 2014

51 Problems in Calculating Limits Using L'Hôpital's Rule with Solutions

(Part 5)

36. $$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)=\lim_{x\to 0}\frac{\sin(x)-x}{x\sin(x)}\overset{LH}{\cong}\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)+x\cos(x)}\overset{LH}{\cong}$$
$$\overset{LH}{\cong}\lim_{x\to 0}\frac{-\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}=\lim_{x\to 0}\frac{-\sin(x)}{2\cos(x)-x\sin(x)}=\frac{-\sin(0)}{2\cos(0)-(0)\sin(0)}=$$
$$=\frac{0}{2(1)-0}=\frac{0}{2}=0.$$

37. $$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2\left(1+\frac{3}{x}\right)}-x\right)=\lim_{x\to\infty}\left(x\sqrt{1+\frac{3}{x}}-x\right)=$$
$$=\lim_{x\to\infty}x\left(\sqrt{1+\frac{3}{x}}-1\right)=\lim_{x\to\infty}\frac{\sqrt{1+\frac{3}{x}}-1}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\left(1+\frac{3}{x}\right)^{\frac{1}{2}}-1}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{(1+3x^{-1})^{\frac{1}{2}}-1}{x^{-1}}\overset{LH}{\cong}$$
$$\overset{LH}{\cong}\lim_{x\to\infty}\frac{\frac{1}{2}(1+3x^{-1})^{\frac{-1}{2}}(-3x^{-2})}{-x^{-2}}=\lim_{x\to\infty}\frac{\left(\frac{1}{2\left(1+\frac{3}{x}\right)^{\frac{1}{2}}}\right)\cdot\left(\frac{-3}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}\frac{\left(\frac{-3}{2x^2\sqrt{1+\frac{3}{x}}}\right)}{\left(\frac{-1}{x^2}\right)}=$$
$$=\lim_{x\to\infty}\left(\frac{-3}{2x^2\sqrt{1+\frac{3}{x}}}\right)\cdot\left(\frac{x^2}{-1}\right)=\lim_{x\to\infty}\frac{3}{2\sqrt{1+\frac{3}{x}}}=\frac{3}{2\sqrt{1+0}}=\frac{3}{2}.$$

38. $\lim_{x\to1^+}\left(\frac{1}{x-1}-\frac{1}{\ln(x)}\right)=\lim_{x\to1^+}\frac{\ln(x)-(x-1)}{\ln(x)(x-1)}=\lim_{x\to1^+}\frac{\ln(x)-x+1}{\ln(x)(x-1)}\overset{LH}{\cong}$

$$\overset{LH}{\cong}\lim_{x\to1^+}\frac{\frac{1}{x}-1}{\frac{1}{x}(x-1)+\ln(x)}=\lim_{x\to1^+}\frac{\left(\frac{1-x}{x}\right)}{\left(\frac{x-1+x\ln(x)}{x}\right)}=\lim_{x\to1^+}\frac{1-x}{x}\cdot\frac{x}{x-1+x\ln(x)}=$$

$$=\lim_{x\to1^+}\frac{1-x}{x-1+x\ln(x)}\overset{LH}{\cong}\lim_{x\to1^+}\frac{-1}{1+\ln(x)+1}=\frac{-1}{2+\ln(1)}=\frac{-1}{2+0}=\frac{-1}{2}.$$

39. $\lim_{x\to\infty}\left(\frac{x^2}{x-1}-\frac{x^2}{x+1}\right)=\lim_{x\to\infty}\frac{x^2(x+1)-x^2(x-1)}{(x-1)(x+1)}=\lim_{x\to\infty}\frac{x^3+x^2-x^3+x^2}{x^2-1}=$

$$=\lim_{x\to\infty}\frac{2x^2}{x^2-1}\overset{LH}{\cong}\lim_{x\to\infty}\frac{4x}{2x}=\lim_{x\to\infty}2=2.$$

40. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=\lim_{x\to\infty}\frac{\sqrt{x^2+x}-\sqrt{x^2-x}}{1}\cdot\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}=$

$$=\lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{x^2+x-x^2+x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x}{(x^2+x)^{\frac{1}{2}}+(x^2-x)^{\frac{1}{2}}}\overset{LH}{\cong}$$

$$\overset{LH}{\cong}\lim_{x\to\infty}\frac{2}{\frac{1}{2}(x^2+x)^{\frac{-1}{2}}(2x+1)+\frac{1}{2}(x^2-x)^{\frac{-1}{2}}(2x-1)}=\lim_{x\to\infty}\frac{2}{\frac{2x+1}{2\sqrt{x^2+x}}+\frac{2x-1}{2\sqrt{x^2-x}}}=$$

$$=\lim_{x\to\infty}\frac{2}{\frac{x\left(2+\frac{1}{x}\right)}{2\sqrt{x^2\left(1+\frac{1}{x}\right)}}+\frac{x\left(2-\frac{1}{x}\right)}{2\sqrt{x^2\left(1-\frac{1}{x}\right)}}}=\lim_{x\to\infty}\frac{2}{\frac{x\left(2+\frac{1}{x}\right)}{2x\sqrt{1+\frac{1}{x}}}+\frac{x\left(2-\frac{1}{x}\right)}{2x\sqrt{1-\frac{1}{x}}}}=$$

$$=\lim_{x\to\infty}\frac{2}{\frac{2+\frac{1}{x}}{2\sqrt{1+\frac{1}{x}}}+\frac{2-\frac{1}{x}}{2\sqrt{1-\frac{1}{x}}}}=\frac{2}{\frac{2+0}{2\sqrt{1+0}}+\frac{2-0}{2\sqrt{1-0}}}=\frac{2}{\frac{2}{2\sqrt{1}}+\frac{2}{2\sqrt{1}}}=\frac{2}{\frac{2}{2}+\frac{2}{2}}=\frac{2}{1+1}=\frac{2}{2}=1.$$