The Weekly Rigor

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51 Problems in Calculating Limits Using L'Hôpital's Rule with Solutions

(Part 6)

41. Let $y = (1+\sin(4x))^{\cot(x)}$.

Hence, $\ln(y) = \ln\left[(1+\sin(4x))^{\cot(x)}\right] = \cot(x)\ln(1+\sin(4x))$.

So, $\lim\limits_{x\to0^+}\ln(y) = \lim\limits_{x\to0^+}\cot(x)\ln(1+\sin(4x)) = \lim\limits_{x\to0^+}\frac{\ln(1+\sin(4x))}{\tan(x)} \overset{LH}{=} \lim\limits_{x\to0^+}\frac{\left(\frac{4\cos(x)}{1+\sin(4x)}\right)}{\sec^2(x)} =$

$= \frac{\left(\frac{4\cos(0)}{1+\sin(0)}\right)}{\sec^2(0)} = \frac{\left(\frac{4(1)}{1+0}\right)}{1} = \frac{\left(\frac{4}{1}\right)}{1} = 4.$ That is, $\lim\limits_{x\to0^+}\ln(y) = 4$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to0^+}(1+\sin(4x))^{\cot(x)} = \lim\limits_{x\to0^+} y =$

$= \lim\limits_{x\to0^+} e^{\ln(y)} = e^{\lim\limits_{x\to0^+}\ln(y)} = e^4$.

42. Let $y = x^x$.

Hence, $\ln(y) = \ln(x^x) = x\ln(x)$.

So, $\lim\limits_{x\to0^+}\ln(y) = \lim\limits_{x\to0^+} x\ln(x) = \lim\limits_{x\to0^+}\frac{\ln(x)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to0^+}\frac{\ln(x)}{x^{-1}} \overset{LH}{=} \lim\limits_{x\to0^+}\frac{\left(\frac{1}{x}\right)}{-x^{-2}} = \lim\limits_{x\to0^+}\frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} =$

$= \lim\limits_{x\to0^+}\frac{1}{x}\cdot\frac{x^2}{-1} = \lim\limits_{x\to0^+} -x = 0.$ That is, $\lim\limits_{x\to0^+}\ln(y) = 0$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to0^+} x^x = \lim\limits_{x\to0^+} y =$

$= \lim\limits_{x\to0^+} e^{\ln(y)} = e^{\lim\limits_{x\to0^+}\ln(y)} = e^0 = 1$.

43. Let $y = (1+x)^{\frac{1}{x}}$.

Hence, $\ln(y) = \ln\left[(1+x)^{\frac{1}{x}}\right] = \frac{1}{x}\ln(1+x) = \dfrac{\ln(1+x)}{x}$.

So, $\lim\limits_{x\to 0}\ln(y) = \lim\limits_{x\to 0}\frac{\ln(1+x)}{x} \overset{LH}{=} \lim\limits_{x\to 0}\frac{1}{1+x} = \frac{1}{1+0} = 1$. That is, $\lim\limits_{x\to 0}\ln(y) = 1$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = \lim\limits_{x\to 0} y =$

$= \lim\limits_{x\to 0} e^{\ln(y)} = e^{\lim_{x\to 0}\ln(y)} = e^1 = e$.

44. Let $y = x^{\frac{1}{1-x}}$.

Hence, $\ln(y) = \ln\left(x^{\frac{1}{1-x}}\right) = \frac{1}{1-x}\ln(x) = \dfrac{\ln(x)}{1-x}$.

So, $\lim\limits_{x\to 1^+}\ln(y) = \lim\limits_{x\to 1^+}\frac{\ln(x)}{1-x} \overset{LH}{=} \lim\limits_{x\to 1^+}\frac{\left(\frac{1}{x}\right)}{-1} = \frac{\left(\frac{1}{1}\right)}{-1} = -1$. That is, $\lim\limits_{x\to 1^+}\ln(y) = -1$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to 1^+} x^{\frac{1}{1-x}} = \lim\limits_{x\to 1^+} y =$

$= \lim\limits_{x\to 1^+} e^{\ln(y)} = e^{\lim_{x\to 1^+}\ln(y)} = e^{-1} = \dfrac{1}{e}$.

45. Let $y = [\ln(x)]^{\frac{1}{x}}$.

Hence, $\ln(y) = \ln\left[[\ln(x)]^{\frac{1}{x}}\right] = \frac{1}{x}\ln[\ln(x)] = \dfrac{\ln[\ln(x)]}{x}$.

So, $\lim\limits_{x\to\infty}\ln(y) = \lim\limits_{x\to\infty}\frac{\ln[\ln(x)]}{x} \overset{LH}{=} \lim\limits_{x\to\infty}\frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{1} = \frac{0}{1} = 0$. That is, $\lim\limits_{x\to\infty}\ln(y) = 0$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to\infty}[\ln(x)]^{\frac{1}{x}} = \lim\limits_{x\to\infty} y =$

$= \lim\limits_{x\to\infty} e^{\ln(y)} = e^{\lim_{x\to\infty}\ln(y)} = e^0 = 1$.