The Weekly Rigor

No. 26 “A mathematician is a machine for turning coffee into theorems.” December 20, 2014

51 Problems in Calculating Limits Using L’Hôpital’s Rule with Solutions

(Part 7)

46. Let $y = x^{\frac{-1}{\ln(x)}}$.

Hence, $\ln(y) = \ln\left(x^{\frac{-1}{\ln(x)}}\right) = \frac{-1}{\ln(x)}\ln(x) = \frac{-\ln(x)}{\ln(x)} = -1$.

So, $\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} (-1) = -1$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim_{x\to 0^+} x^{\frac{-1}{\ln(x)}} = \lim_{x\to 0^+} y =$

$= \lim_{x\to 0^+} e^{\ln(y)} = e^{\lim_{x\to 0^+} \ln(y)} = e^{-1} = \frac{1}{e}$.

47. Let $y = (1+2x)^{\frac{1}{2\ln(x)}}$.

Hence, $\ln(y) = \ln\left[(1+2x)^{\frac{1}{2\ln(x)}}\right] = \frac{1}{2\ln(x)}\ln(1+2x) = \frac{\ln(1+2x)}{2\ln(x)}$.

So, $\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln(x)} \overset{LH}{=} \lim_{x\to\infty} \frac{\left(\frac{2}{1+2x}\right)}{\left(\frac{2}{x}\right)} = \lim_{x\to\infty} \frac{2}{1+2x}\cdot\frac{x}{2} = \lim_{x\to\infty} \frac{x}{1+2x} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$.

That is, $\lim_{x\to\infty} \ln(y) = \frac{1}{2}$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim_{x\to\infty} (1+2x)^{\frac{1}{2\ln(x)}} = \lim_{x\to\infty} y =$

$= \lim_{x\to\infty} e^{\ln(y)} = e^{\lim_{x\to\infty} \ln(y)} = e^{\frac{1}{2}}$.

48. Let $y = (e^x + x)^{\frac{2}{x}}$.

Hence, $\ln(y) = \ln\left[(e^x + x)^{\frac{2}{x}}\right] = \frac{2}{x}\ln(e^x + x) = \dfrac{2\ln(e^x + x)}{x}$.

So, $\lim\limits_{x\to 0}\ln(y) = \lim\limits_{x\to 0}\dfrac{2\ln(e^x + x)}{x} \overset{LH}{\cong} \lim\limits_{x\to 0}\dfrac{\left(\frac{2[e^x+1]}{e^x+x}\right)}{1} = \dfrac{2(e^0+1)}{e^0+0} = \dfrac{2(1+1)}{1} = 2\cdot 2 = 4.$

That is, $\lim\limits_{x\to 0}\ln(y) = 4$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{x\to 0}(e^x + x)^{\frac{2}{x}} = \lim\limits_{x\to 0} y =$

$= \lim\limits_{x\to 0} e^{\ln(y)} = e^{\lim\limits_{x\to 0}\ln(y)} = e^4$.

49. Let $y = (1 + hx)^{\frac{1}{h}}$.

Hence, $\ln(y) = \ln\left[(1 + hx)^{\frac{1}{h}}\right] = \frac{1}{h}\ln(1 + hx) = \dfrac{\ln(1 + hx)}{h}$.

So, $\lim\limits_{h\to 0}\ln(y) = \lim\limits_{h\to 0}\dfrac{\ln(1 + hx)}{h} \overset{LH}{\cong} \lim\limits_{h\to 0}\dfrac{\left(\frac{x}{1+hx}\right)}{1} = \dfrac{\left(\frac{x}{1+0}\right)}{1} = \dfrac{x}{1+0} = x.$

That is, $\lim\limits_{h\to 0}\ln(y) = x$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{h\to 0}(1 + hx)^{\frac{1}{h}} = \lim\limits_{h\to 0} y =$

$= \lim\limits_{h\to 0} e^{\ln(y)} = e^{\lim\limits_{h\to 0}\ln(y)} = e^x$.