The Weekly Rigor

No. 27 “A mathematician is a machine for turning coffee into theorems.” December 27, 2014

51 Problems in Calculating Limits Using L’Hôpital’s Rule with Solutions

(Part 8)

50. Let $y = \left(1+\frac{x}{n}\right)^n$.

Hence, $\ln(y) = \ln\left[\left(1+\frac{x}{n}\right)^n\right] = n\ln\left(1+\frac{x}{n}\right) = \dfrac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$.

So, $\lim\limits_{n\to\infty}\ln(y) = \lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = \lim\limits_{n\to\infty}\frac{\ln\left(\frac{n+x}{n}\right)}{\frac{1}{n}} = \lim\limits_{n\to\infty}\frac{\ln(n+x)-\ln(n)}{n^{-1}} \overset{LH}{=} \lim\limits_{n\to\infty}\frac{\left(\frac{1}{n+x}-\frac{1}{n}\right)}{-n^{-2}} = \lim\limits_{n\to\infty}\frac{\left(\frac{n-n-x}{n(n+x)}\right)}{\left(\frac{-1}{n^2}\right)} =$

$= \lim\limits_{n\to\infty}\dfrac{-x}{n(n+x)}\cdot\dfrac{n^2}{-1} = \lim\limits_{n\to\infty}\dfrac{nx}{n+x} = \lim\limits_{n\to\infty}\dfrac{nx}{n\left(1+\frac{x}{n}\right)} = \lim\limits_{n\to\infty}\dfrac{x}{\left(1+\frac{x}{n}\right)} = \dfrac{x}{1+0} = x.$

That is, $\lim\limits_{n\to\infty}\ln(y) = x$.

Now $y = y \iff \ln(y) = \ln(y) \iff e^{\ln(y)} = y$. Therefore, $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim\limits_{n\to\infty} y =$

$= \lim\limits_{n\to\infty} e^{\ln(y)} = e^{\lim\limits_{n\to\infty}\ln(y)} = e^x.$

51. Let $y=\left(1+\frac{r}{n}\right)^{nt}$.

Hence, $\ln(y)=\ln\left[\left(1+\frac{r}{n}\right)^{nt}\right]=nt\ln\left(1+\frac{r}{n}\right)=\dfrac{\text{tln}\left(1+\frac{r}{n}\right)}{\frac{1}{n}}=\dfrac{\text{tln}\left(\frac{n+r}{n}\right)}{\frac{1}{n}}=$

$=\dfrac{t\ln(n+r)-t\ln(n)}{\dfrac{1}{n}}.$

So, $\lim\limits_{n\to\infty}\ln(y)=\lim\limits_{n\to\infty}\frac{t\ln(n+r)-t\ln(n)}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{t\ln(n+r)-t\ln(n)}{n^{-1}}\overset{LH}{\cong}\lim\limits_{n\to\infty}\frac{\frac{t}{n+r}-\frac{t}{n}}{-n^{-2}}=\lim\limits_{n\to\infty}\frac{\frac{t}{n+r}-\frac{t}{n}}{\frac{-1}{n^2}}=$

$=\lim\limits_{n\to\infty}\left(\dfrac{-tn^2}{n+r}+\dfrac{tn^2}{n}\right)=\lim\limits_{n\to\infty}\dfrac{-tn^3+tn^3+rtn^2}{n(n+r)}=\lim\limits_{n\to\infty}\dfrac{rtn}{n+r}=\lim\limits_{n\to\infty}\dfrac{rtn}{n\left(1+\frac{r}{n}\right)}=$

$=\lim\limits_{n\to\infty}\dfrac{rt}{1+\frac{r}{n}}=\dfrac{rt}{1+0}=rt.$

That is, $\lim\limits_{n\to\infty}\ln(y)=rt$.

Now $y=y\iff\ln(y)=\ln(y)\iff e^{\ln(y)}=y$. Therefore, $\lim\limits_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}=\lim\limits_{n\to\infty}y=$

$=\lim\limits_{n\to\infty}e^{\ln(y)}=e^{\lim\limits_{n\to\infty}\ln(y)}=e^{rt}.$