The Weekly Rigor

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51 Problems in Calculating Limits Using L'Hôpital's Rule with Solutions
(Part 10)

SOME BACKGROUND MATHEMATICAL FACTS USEFUL IN CALCULATING LIMITS USING L'HÔPITAL'S RULE

$e^0 = 1$ $\qquad \ln(1) = 0$ $\qquad \ln(e) = 1$

$\sin(0) = 0$ $\qquad \cos(0) = 1$ $\qquad \tan(0) = 0$

$\sin\left(\frac{\pi}{2}\right) = 1$ $\qquad \cos\left(\frac{\pi}{2}\right) = 0$ $\qquad \cot\left(\frac{\pi}{2}\right) = 0$

$\sin(\pi) = 0$ $\qquad \cos(\pi) = -1$ $\qquad \tan(\pi) = 0$

$\lim_{x\to\infty} \frac{1}{x} = 0$ $\qquad a = \frac{1}{\left(\frac{1}{a}\right)}$

$\ln(ab) = \ln(a) + \ln(b)$ $\qquad \ln(a^c) = c\ln(a)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ $\qquad e^{\ln(x)} = x$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ $\qquad \cot(x) = \frac{\cos(x)}{\sin(x)}$

$\sec(x) = \frac{1}{\cos(x)}$ $\qquad \csc(x) = \frac{1}{\sin(x)}$ $\qquad \cot(x) = \frac{1}{\tan(x)}$

$[e^x]' = e^x$ $\qquad [\sin(x)]' = \cos(x)$ $\qquad [\sec(x)]' = \sec(x)\tan(x)$

$[\ln(x)]' = \frac{1}{x}$ $\qquad [\cos(x)]' = -\sin(x)$ $\qquad [\csc(x)]' = -\csc(x)\cot(x)$

$[a^x]' = a^x\ln(a)$ $\qquad [\tan(x)]' = \sec^2(x)$ $\qquad [\cot(x)]' = -\csc^2(x)$

TWO APPLICATIONS OF L'HÔPITAL'S RULE TO COMPARING
THE RATES OF GROWTH OF THREE FUNCTIONS

Theorem 1: $\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$ for any integer *n*.

Preliminary Remark: This theorem shows that the exponential function approaches infinity faster than any power of *x*.

Proof: $\lim_{x\to\infty}\frac{e^x}{x^n}\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x}{nx^{n-1}}\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}\overset{LH}{\cong}\cdots\overset{LH}{\cong}\lim_{x\to\infty}\frac{e^x}{n!}=\infty.$

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Theorem 2: $\lim_{x\to\infty}\frac{\ln(x)}{x^p}=0$ for any number $p>0$.

Preliminary Remark: This theorem shows that the logarithmic function approaches infinity more slowly than any power of *x*.

Proof: $\lim_{x\to\infty}\frac{\ln(x)}{x^p}\overset{LH}{\cong}\lim_{x\to\infty}\frac{\frac{1}{x}}{px^{p-1}}=\lim_{x\to\infty}\frac{1}{x}\cdot\frac{1}{px^{p-1}}=\lim_{x\to\infty}\frac{1}{px^p}=0.$

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