# The Weekly Rigor

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## 101 Problems in Calculating Trigonometric Limits with Solutions

(Part 1)

### INTRODUCTION

One of the early challenges for the beginning calculus student is computing trigonometric limits. I hope that working through the problems in the following issues will make such a student a master of the topic.

There are three fundamental math facts the student must automatize before starting on the problems:

1. $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$ 2. $\lim_{\theta\to 0}\sin(\theta)=0$ 3. $\lim_{\theta\to 0}\cos(\theta)=1$

The chances are very good that the student enjoyed a thirty-minute lecture in class where the first fact was proved using the Sandwich Theorem (also known as the Squeeze Theorem). The student may safely forget the proof, but not the result, for that fact shows up in the vast majority of trigonometric limits. The two other facts are true because sine and cosine are continuous functions and $\sin(0)=0$ and $\cos(0)=1$.

The second set of math facts the student will need are some of the trigonometric identities:

$$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta)=\frac{1}{\cos(\theta)} \qquad \csc(\theta)=\frac{1}{\sin(\theta)} \qquad \cot(\theta)=\frac{1}{\tan(\theta)}$$

$$\sin^2(\theta)+\cos^2(\theta)=1$$

The final mathematical prerequisite concerns fractions. There is a crucial fact about fractions that comes up in almost every trigonometric limit problem: We multiply fractions by multiplying the numerators and denominators. Hence, the order of the factors, say, in the numerator, can be whatever we please. For example, if we wish to multiply

$$\frac{2}{3}\cdot\frac{5}{7}$$

we could, if we desire, modify the fractions to look like

$$\frac{5}{3}\cdot\frac{2}{7}.$$

In either case, the multiplication would result in the answer

$$\frac{10}{21}$$

Although this sort of manipulation of factor order is not usually important in ordinary arithmetic, the solution to a trigonometric limit often requires manipulating the order of the factors to fit certain arrangements that we can compute the limits of. For example, in the problem

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{1}\cdot\frac{\cos(\theta)}{\theta}$$

it is helpful to rewrite the problem as

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\cos(\theta)}{1}$$

so that we may apply the limit laws and the basic trigonometric limits, viz.,

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\cos(\theta)}{1}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\cos(\theta)=(1)(1)=1.$$

The reader will see this issue come up repeatedly in the solutions to the problems.

I have organized the problems into groups. Problems 1-11 I call the "Sine Group," problems 12-18 the "Cosine Group," problems 19-28 the "Cosine-Conjugate Group," problems 29-36 the "Tangent Group," problems 37-46 the "Trig-Identity Heavy Group," and problems 47-101 the "Miscellaneous Group." This grouping will help the student see some common characteristics among many of the problems. In the "Trig-Identity Heavy Group" there are some problems that use advanced trigonometric identities, such as the addition/subtraction formulas.

Occasionally in the solutions I use a designation such as "$\overset{\#1}{\triangleq}$" to point the reader back to an earlier solution.