The Weekly Rigor

No. 33 "A mathematician is a machine for turning coffee into theorems." February 7, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 4)

SOLUTIONS

Sine Group:

1. $\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} = \lim_{\theta\to 0}\frac{1}{\left(\frac{\sin(\theta)}{\theta}\right)} = \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right)} = \frac{1}{1} = 1.$

2. $\lim_{\theta\to 0}\frac{3\sin(\theta)}{\theta} = \lim_{\theta\to 0} 3\cdot\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0} 3\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 3\cdot 1 = 3.$

3. $\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{3}{1} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0} 3 = 1\cdot 3 = 3.$

Remark: $3\theta \to 0$ as $\theta \to 0$. That is why we can say $\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta} = 1$. Similar cases will arise in some of the following solutions.

4. $\lim_{\theta\to 0}\frac{\sin(3\theta)}{\sin(4\theta)} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{1}\cdot\frac{1}{\sin(4\theta)} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{1}\cdot\frac{3\theta}{3\theta}\cdot\frac{1}{\sin(4\theta)}\cdot\frac{4\theta}{4\theta} =$

$= \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{3\theta}{1}\cdot\frac{4\theta}{\sin(4\theta)}\cdot\frac{1}{4\theta} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{4\theta}{\sin(4\theta)}\cdot\frac{3\theta}{4\theta} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{4\theta}{\sin(4\theta)}\cdot\frac{3}{4} =$

$= \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{4\theta}{\sin(4\theta)}\cdot\lim_{\theta\to 0}\frac{3}{4} \overset{\#1}{=} 1\cdot 1\cdot\frac{3}{4} = \frac{3}{4}.$

5. $\lim_{\theta\to 0^+}\frac{\theta}{\sin(\sqrt{\theta})} = \lim_{\theta\to 0^+}\frac{\sqrt{\theta}}{\sin(\sqrt{\theta})}\cdot\frac{\sqrt{\theta}}{1} = \lim_{\theta\to 0^+}\frac{\sqrt{\theta}}{\sin(\sqrt{\theta})}\cdot\lim_{\theta\to 0^+}\sqrt{\theta} = \lim_{\theta\to 0^+}\frac{\sqrt{\theta}}{\sin(\sqrt{\theta})}\cdot\sqrt{\lim_{\theta\to 0^+}\theta} \overset{\#1}{=}$

$\overset{\#1}{=} 1\cdot\sqrt{0} = 1\cdot 0 = 0.$

6. $\lim_{\theta\to 0^+}\frac{\sin(\theta)}{\sqrt{\theta}} = \lim_{\theta\to 0^+}\frac{\sin(\theta)}{\sqrt{\theta}}\cdot\frac{\sqrt{\theta}}{\sqrt{\theta}} = \lim_{\theta\to 0^+}\frac{\sin(\theta)}{\theta}\cdot\frac{\sqrt{\theta}}{1} = \lim_{\theta\to 0^+}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0^+}\sqrt{\theta} =$

$= \lim_{\theta\to 0^+}\frac{\sin(\theta)}{\theta}\cdot\sqrt{\lim_{\theta\to 0^+}\theta} = 1\cdot\sqrt{0} = 1\cdot 0 = 0.$

7. $\lim_{\theta\to 0}\theta\csc(\theta) = \lim_{\theta\to 0}\theta\cdot\frac{1}{\sin(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} \overset{\#1}{=} 1.$

8. $\lim_{\theta\to 0}\frac{\sin(-3\theta)}{4\theta} = \lim_{\theta\to 0}\frac{-\sin(3\theta)}{4\theta} = \lim_{\theta\to 0}\frac{-1}{4}\cdot\frac{\sin(3\theta)}{\theta} = -\frac{1}{4}\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta} =$

$= -\frac{1}{4}\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3} = -\frac{3}{4}\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta} = -\frac{3}{4}(1) = -\frac{3}{4}.$

Alternate Solution:

$\lim_{\theta\to 0}\frac{\sin(-3\theta)}{4\theta} = \frac{1}{4}\lim_{\theta\to 0}\frac{\sin(-3\theta)}{\theta} = \frac{1}{4}\lim_{\theta\to 0}\frac{\sin(-3\theta)}{\theta}\cdot\frac{-3}{-3} = \frac{-3}{4}\lim_{\theta\to 0}\frac{\sin(-3\theta)}{-3\theta} =$

$= \frac{-3}{4}\cdot(1) = \frac{-3}{4}.$

Let a and b be nonzero numbers.

9. $\lim_{\theta\to 0}\frac{\sin(a\theta)}{b\theta} = \lim_{\theta\to 0}\frac{1}{b}\cdot\frac{\sin(\theta)}{\theta} = \frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta} = \frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{a}{a} = \frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}.$

$= \frac{a}{b}\cdot 1 = \frac{a}{b}.$

Let a and b be nonzero numbers.

10. $\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)} = \lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{1}{\sin(b\theta)} = \lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{1}{\sin(b\theta)}\cdot\frac{ab\theta}{ab\theta} =$

$= \frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{b\theta}{\sin(b\theta)} = \frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)} \overset{\#1}{=} \frac{a}{b}\cdot 1\cdot 1 = \frac{a}{b}.$

Let k be a nonzero number.

11. $\lim_{\theta\to 0}\frac{\sin^2(k\theta)}{\theta^2} = \lim_{\theta\to 0}\frac{\sin(k\theta)}{\theta}\cdot\frac{\sin(k\theta)}{\theta} = \lim_{\theta\to 0}\frac{\sin(k\theta)}{\theta}\cdot\frac{\sin(k\theta)}{\theta}\cdot\frac{k^2}{k^2} =$

$= \lim_{\theta\to 0}\frac{\sin(k\theta)}{k\theta}\cdot\frac{\sin(k\theta)}{k\theta}\cdot\frac{k^2}{1} = \lim_{\theta\to 0}\frac{\sin(k\theta)}{k\theta}\cdot\lim_{\theta\to 0}\frac{\sin(k\theta)}{k\theta}\cdot\lim_{\theta\to 0}\frac{k^2}{1} =$

$= \lim_{\theta\to 0}\frac{\sin(k\theta)}{k\theta}\cdot\lim_{\theta\to 0}\frac{\sin(k\theta)}{k\theta}\cdot k^2 = 1\cdot 1\cdot k^2 = k^2.$