The Weekly Rigor

No. 34 "A mathematician is a machine for turning coffee into theorems." February 14, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 5)

Cosine Group:

12. $\lim_{\theta\to 0}\sec(\theta) = \lim_{\theta\to 0}\frac{1}{\cos(\theta)} = \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0}\cos(\theta)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$

13. $\lim_{\theta\to 0} 3\cos(\theta) = \lim_{\theta\to 0} 3 \cdot \lim_{\theta\to 0}\cos(\theta) = 3\cos(0) = 3\cdot 1 = 3.$

14. $\lim_{\theta\to 0}\frac{1}{\cos(\theta)} = \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0}\cos(\theta)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$

15. $\lim_{\theta\to 0}\frac{\cos(\theta)}{\theta} =$ Does not exist.

Remark: Note that the numerator is approaching the fixed number 1 while the denominator is approaching 0. Furthermore, $\lim_{\theta\to 0^+}\frac{\cos(\theta)}{\theta} = +\infty$ while $\lim_{\theta\to 0^-}\frac{\cos(\theta)}{\theta} = -\infty$.

16. $\lim_{\theta\to 0}\frac{\theta}{\cos(\theta)} = \frac{\lim_{\theta\to 0}\theta}{\lim_{\theta\to 0}\cos(\theta)} = \frac{0}{\cos(0)} = \frac{0}{1} = 0.$

17. $\lim_{\theta\to 0}\frac{\cos(3\theta)}{\cos(4\theta)} = \frac{\lim_{\theta\to 0}\cos(3\theta)}{\lim_{\theta\to 0}\cos(4\theta)} = \frac{\cos(0)}{\cos(0)} = \frac{1}{1} = 1.$

18. $\lim_{\theta\to 0}\frac{\cos(a\theta)}{\cos(b\theta)} = \frac{\lim_{\theta\to 0}\cos(a\theta)}{\lim_{\theta\to 0}\cos(b\theta)} = \frac{\cos(0)}{\cos(0)} = \frac{1}{1} = 1.$

Cosine-Conjugate Group:

19. $$\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\theta[1+\cos(\theta)]} = \lim_{\theta\to 0}\frac{\sin^2(\theta)}{\theta[1+\cos(\theta)]} =$$
$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{1}\cdot\frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{1}\cdot\frac{\theta}{\theta}\cdot\frac{1}{1+\cos(\theta)} =$$
$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta}\cdot\frac{\theta}{1}\cdot\frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{1}\cdot\lim_{\theta\to 0}\frac{1}{1+\cos(\theta)}$$
$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{1}\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[1+\cos(\theta)]} = 1\cdot 1\cdot 0\cdot\frac{1}{1+\cos(0)} = \frac{0}{1+1} = 0.$$

20. $$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1} = \lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta[\cos(\theta)+1]} = \lim_{\theta\to 0}\frac{-[1-\cos^2(\theta)]}{\theta[\cos(\theta)+1]} =$$
$$= \lim_{\theta\to 0}\frac{-\sin^2(\theta)}{\theta[\cos(\theta)+1]} = -\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{1}\cdot\frac{1}{\cos(\theta)+1} =$$
$$= -\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{1}\cdot\frac{\theta}{\theta}\cdot\frac{1}{\cos(\theta)+1} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta}\cdot\frac{\theta}{1}\cdot\frac{1}{\cos(\theta)+1} =$$
$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{1}\cdot\lim_{\theta\to 0}\frac{1}{\cos(\theta)+1} = 1\cdot 1\cdot 0\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[\cos(\theta)+1]} =$$
$$= 1\cdot 1\cdot 0\cdot\frac{1}{\cos(0)+1} = \frac{0}{1+1} = 0.$$

21. $$\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta^{\frac{2}{3}}} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta^{\frac{2}{3}}}\cdot\frac{\theta^{\frac{1}{3}}}{\theta^{\frac{1}{3}}} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}\cdot\theta^{\frac{1}{3}} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}\cdot\lim_{\theta\to 0}\theta^{\frac{1}{3}} \overset{\#19}{\cong}$$
$$\overset{\#19}{\cong} 0\cdot 0 = 0.$$

22. $$\lim_{\theta\to 0}\frac{2\cos(\theta)-2}{3\theta} = \lim_{\theta\to 0}\frac{2(\cos(\theta)-1)}{3\theta} = \lim_{\theta\to 0}\frac{2}{3}\cdot\frac{\cos(\theta)-1}{\theta} = \frac{2}{3}\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} \overset{\#20}{\cong} \frac{2}{3}\cdot 0 = 0.$$