The Weekly Rigor

No. 35 “A mathematician is a machine for turning coffee into theorems.” February 21, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 6)

23. $$\lim_{\theta\to 0}\frac{\theta^2}{1-\cos(\theta)}=\lim_{\theta\to 0}\frac{\theta}{1}\cdot\frac{\theta}{1-\cos(\theta)}=\lim_{\theta\to 0}\frac{\theta}{1}\cdot\frac{\theta}{1-\cos(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\theta}{1}\cdot\frac{\theta[1+\cos(\theta)]}{1-\cos^2\theta}=\lim_{\theta\to 0}\frac{\theta}{1}\cdot\frac{\theta[1+\cos(\theta)]}{\sin^2\theta}=\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{[1+\cos(\theta)]}{1}=$$
$$=\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{[1+\cos(\theta)]}{1}=\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{\lim_{\theta\to 0}[1+\cos(\theta)]}{\lim_{\theta\to 0}1}\overset{\#1}{\cong}$$
$$\overset{\#1}{\cong}1\cdot 1\cdot\frac{1+\cos(0)}{1}=\frac{1+1}{1}=2.$$

24. $$\lim_{\theta\to 0}\frac{1-\cos(5\theta)}{\cos(7\theta)-1}=\lim_{\theta\to 0}\frac{1-\cos(5\theta)}{-[1-\cos(7\theta)]}=-\lim_{\theta\to 0}\frac{1-\cos(5\theta)}{1-\cos(7\theta)}\cdot\frac{1+\cos(5\theta)}{1+\cos(5\theta)}\cdot\frac{1+\cos(7\theta)}{1+\cos(7\theta)}=$$
$$=-\lim_{\theta\to 0}\frac{1-\cos^2(5\theta)}{1-\cos^2(7\theta)}\cdot\frac{1+\cos(7\theta)}{1+\cos(5\theta)}=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\frac{1+\cos(7\theta)}{1+\cos(5\theta)}=$$
$$=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\lim_{\theta\to 0}\frac{1+\cos(7\theta)}{1+\cos(5\theta)}=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\frac{\lim_{\theta\to 0}[1+\cos(7\theta)]}{\lim_{\theta\to 0}[1+\cos(5\theta)]}=$$
$$-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\frac{1+\cos(0)}{1+\cos(0)}=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\frac{1+1}{1+1}=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}\cdot\frac{2}{2}=$$
$$=-\lim_{\theta\to 0}\frac{\sin^2(5\theta)}{\sin^2(7\theta)}=-\lim_{\theta\to 0}\frac{\sin(5\theta)}{\sin(7\theta)}\cdot\frac{\sin(5\theta)}{\sin(7\theta)}=$$
$$=-\lim_{\theta\to 0}\frac{\sin(5\theta)}{1}\cdot\frac{\sin(5\theta)}{1}\cdot\frac{1}{\sin(7\theta)}\cdot\frac{1}{\sin(7\theta)}\cdot\frac{25\theta^2}{25\theta^2}\cdot\frac{49\theta^2}{49\theta^2}=$$
$$=-\lim_{\theta\to 0}\frac{\sin(5\theta)}{5\theta}\cdot\frac{\sin(5\theta)}{5\theta}\cdot\frac{7\theta}{\sin(7\theta)}\cdot\frac{7\theta}{\sin(7\theta)}\cdot\frac{25\theta^2}{49\theta^2}=$$
$$=-\lim_{\theta\to 0}\frac{\sin(5\theta)}{5\theta}\cdot\lim_{\theta\to 0}\frac{\sin(5\theta)}{5\theta}\cdot\lim_{\theta\to 0}\frac{7\theta}{\sin(7\theta)}\cdot\lim_{\theta\to 0}\frac{7\theta}{\sin(7\theta)}\cdot\lim_{\theta\to 0}\frac{25\theta^2}{49\theta^2}\overset{\#1}{\cong}$$
$$\overset{\#1}{\cong}-\left(1\cdot 1\cdot 1\cdot 1\cdot\frac{25}{49}\right)=-\frac{25}{49}$$

25. $\lim_{\theta \to 0} \frac{1 - 2\theta^2 - 2\cos(\theta) + \cos^2(\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{1 - 2\theta^2 - 2\cos(\theta) + 1 - \sin^2(\theta)}{\theta^2} =$

$= \lim_{\theta \to 0} \frac{2 - 2\theta^2 - 2\cos(\theta) - \sin^2(\theta)}{\theta^2} = \lim_{\theta \to 0} \left(\frac{2 - 2\cos(\theta)}{\theta^2} - \frac{2\theta^2 + \sin^2(\theta)}{\theta^2} \right) =$

$= \lim_{\theta \to 0} \frac{2[1 - \cos(\theta)]}{\theta^2} - \lim_{\theta \to 0} \frac{2\theta^2 + \sin^2(\theta)}{\theta^2} = 2 \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} - \lim_{\theta \to 0} \frac{2\theta^2 + \sin^2(\theta)}{\theta^2} =$

$= 2 \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} - \lim_{\theta \to 0} \left(2 + \frac{\sin^2(\theta)}{\theta^2} \right) =$

$= 2 \lim_{\theta \to 0} \frac{1 - \cos^2(\theta)}{\theta^2(1 + \cos(\theta))} - \lim_{\theta \to 0} 2 - \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2} = 2 \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2(1 + \cos(\theta))} - 2 - 1 =$

$= 2 \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2} \cdot \frac{1}{1 + \cos(\theta)} - 3 = 2 \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2} \cdot \lim_{\theta \to 0} \frac{1}{1 + \cos(\theta)} - 3 =$

$= 2 \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} [1 + \cos(\theta)]} - 3 = 2 \left(1 \cdot 1 \cdot \frac{1}{1 + \cos(0)} \right) - 3 =$

$= 2 \left(\frac{1}{1 + 1} \right) - 3 = 2 \left(\frac{1}{2} \right) - 3 = 1 - 3 = -2.$

26. $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta^2} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} = \lim_{\theta \to 0} \frac{1 - \cos^2(\theta)}{\theta^2[1 + \cos(\theta)]} =$

$= \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta^2(1 + \cos(\theta))} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\theta} \cdot \frac{1}{1 + \cos(\theta)} =$

$= \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{1 + \cos(\theta)} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} [1 + \cos(\theta)]} =$

$= 1 \cdot 1 \cdot \frac{1}{1 + \cos(0)} = \frac{1}{1 + 1} = \frac{1}{2}.$