The Weekly Rigor

No. 36 "A mathematician is a machine for turning coffee into theorems." February 28, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 7)

27. $$\lim_{\theta\to 0}\frac{1-\cos(3\theta)}{\theta^2}=\lim_{\theta\to 0}\frac{1-\cos(3\theta)}{\theta^2}\cdot\frac{1+\cos(3\theta)}{1+\cos(3\theta)}=\lim_{\theta\to 0}\frac{1-\cos^2(3\theta)}{\theta^2(1+\cos(3\theta))}=$$
$$=\lim_{\theta\to 0}\frac{\sin^2(3\theta)}{\theta^2[1+\cos(3\theta)]}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{\sin(3\theta)}{\theta}\cdot\frac{1}{1+\cos(3\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3}\cdot\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3}\cdot\frac{1}{1+\cos(3\theta)}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{\sin(3\theta)}{3\theta}\cdot\frac{9}{1+\cos(3\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{9}{1+\cos(3\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{\lim_{\theta\to 0}9}{\lim_{\theta\to 0}[1+\cos(3\theta)]}=1\cdot 1\cdot\frac{9}{1+\cos(0)}=\frac{9}{1+1}=\frac{9}{2}.$$

Let a and b be nonzero numbers.

28. $$\lim_{\theta\to 0}\frac{1-\cos(a\theta)}{b\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{1-\cos(a\theta)}{\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{1-\cos(a\theta)}{\theta}\cdot\frac{1+\cos(a\theta)}{1+\cos(a\theta)}=$$
$$=\frac{1}{b}\lim_{\theta\to 0}\frac{1-\cos^2(a\theta)}{\theta[1+\cos(a\theta)]}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin^2(a\theta)}{\theta[1+\cos(a\theta)]}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{\sin(a\theta)}{1}\cdot\frac{1}{1+\cos(a\theta)}=$$
$$=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{a}{a}\cdot\frac{\sin(a\theta)}{1}\cdot\frac{1}{1+\cos(a\theta)}=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{\sin(a\theta)}{1}\cdot\frac{1}{1+\cos(a\theta)}=$$
$$=\frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\sin(a\theta)\cdot\lim_{\theta\to 0}\frac{1}{1+\cos(a\theta)}=$$
$$=\frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\sin(a\theta)\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[1+\cos(a\theta)]}=\frac{a}{b}\cdot 1\cdot 0\cdot\frac{1}{1+1}=0.$$

Tangent Group:

29. $\lim_{\theta\to 0}\tan(\theta) = \lim_{\theta\to 0}\frac{\sin(\theta)}{\cos(\theta)} = \frac{\lim_{\theta\to 0}\sin(\theta)}{\lim_{\theta\to 0}\cos(\theta)} = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0.$

30. $\lim_{\theta\to 0}3\tan(\theta) = \lim_{\theta\to 0}\frac{3\sin(\theta)}{\cos(\theta)} = \frac{\lim_{\theta\to 0}3\sin(\theta)}{\lim_{\theta\to 0}\cos(\theta)} = \frac{3\lim_{\theta\to 0}\sin(\theta)}{\lim_{\theta\to 0}\cos(\theta)} = \frac{3\sin(0)}{\cos(0)} = \frac{3\cdot 0}{1} = 0.$

31. $\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta} = \lim_{\theta\to 0}\frac{\left(\frac{\sin(\theta)}{\cos(\theta)}\right)}{\left(\frac{\theta}{1}\right)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{1}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{1}{\cos(\theta)} =$
$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos(\theta)} = 1\cdot\frac{1}{\cos(0)} = \frac{1}{1} = 1.$

32. $\lim_{\theta\to 0}\frac{\theta}{\tan(\theta)} = \lim_{\theta\to 0}\frac{1}{\left(\frac{\tan(\theta)}{\theta}\right)} = \frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\left(\frac{\tan(\theta)}{\theta}\right)} \overset{\#31}{=} \frac{1}{1} = 1.$

33. $\lim_{\theta\to 0}\frac{\tan(3\theta)}{\theta} = \lim_{\theta\to 0}\frac{\tan(3\theta)}{\theta}\cdot\frac{3}{3} = \lim_{\theta\to 0}\frac{\tan(3\theta)}{3\theta}\cdot\frac{3}{1} = \lim_{\theta\to 0}\frac{\tan(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}3 = \lim_{\theta\to 0}\frac{\tan(3\theta)}{3\theta}\cdot 3 \overset{\#31}{=}$
$\overset{\#31}{=} 1\cdot 3 = 3.$

34. $\lim_{\theta\to 0}\theta\cot(\theta) = \lim_{\theta\to 0}\frac{\theta}{\tan(\theta)} \overset{\#32}{=} 1.$