The Weekly Rigor

No. 37 “A mathematician is a machine for turning coffee into theorems.” March 7, 2015

101 Problems in Calculating Trigonometric Limits with Solutions
(Part 8)

35. $$\lim_{\theta\to 0}\frac{\tan(3\theta)}{\tan(4\theta)}=\lim_{\theta\to 0}\frac{\left(\frac{\sin(3\theta)}{\cos(3\theta)}\right)}{\left(\frac{\sin(4\theta)}{\cos(4\theta)}\right)}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{\cos(3\theta)}\cdot\frac{\cos(4\theta)}{\sin(4\theta)}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{1}\cdot\frac{1}{\sin(4\theta)}\cdot\frac{\cos(4\theta)}{\cos(3\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{1}\cdot\frac{3\theta}{3\theta}\cdot\frac{1}{\sin(4\theta)}\cdot\frac{4\theta}{4\theta}\cdot\frac{\cos(4\theta)}{\cos(3\theta)}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{4\theta}{\sin(4\theta)}\cdot\frac{3\theta}{4\theta}\cdot\frac{\cos(4\theta)}{\cos(3\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{4\theta}{\sin(4\theta)}\cdot\lim_{\theta\to 0}\frac{3}{4}\cdot\lim_{\theta\to 0}\frac{\cos(4\theta)}{\cos(3\theta)}=1\cdot 1\cdot\frac{3}{4}\cdot\frac{\lim_{\theta\to 0}\cos(4\theta)}{\lim_{\theta\to 0}\cos(3\theta)}=\frac{3}{4}\cdot\frac{\cos(0)}{\cos(0)}$$
$$=\frac{3}{4}\cdot\frac{1}{1}=\frac{3}{4}.$$

Alternate Solution:

$$\lim_{\theta\to 0}\frac{\tan(3\theta)}{\tan(4\theta)}=\lim_{\theta\to 0}\frac{\tan(3\theta)}{1}\cdot\frac{1}{\tan(4\theta)}=\lim_{\theta\to 0}\frac{\tan(3\theta)}{3\theta}\cdot\frac{4\theta}{\tan(4\theta)}\cdot\frac{3}{4}=$$
$$=\lim_{\theta\to 0}\frac{\tan(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{4\theta}{\tan(4\theta)}\cdot\lim_{\theta\to 0}\frac{3}{4}\overset{\#31\ \&\ \#32}{=}1\cdot 1\cdot\frac{3}{4}=\frac{3}{4}.$$

Let a and b be nonzero numbers.

36. $$\lim_{\theta\to 0}\frac{\tan(a\theta)}{\tan(b\theta)}=\lim_{\theta\to 0}\frac{\left(\frac{\sin(a\theta)}{\cos(b\theta)}\right)}{\left(\frac{\sin(a\theta)}{\cos(b\theta)}\right)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\cos(a\theta)}\cdot\frac{\cos(b\theta)}{\sin(b\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{1}{\sin(b\theta)}\cdot\frac{\cos(b\theta)}{\cos(a\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{a\theta}{a\theta}\cdot\frac{1}{\sin(b\theta)}\cdot\frac{b\theta}{b\theta}\cdot\frac{\cos(b\theta)}{\cos(a\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{b\theta}{\sin(b\theta)}\cdot\frac{a\theta}{b\theta}\cdot\frac{\cos(b\theta)}{\cos(a\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)}\cdot\lim_{\theta\to 0}\frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\cos(b\theta)}{\cos(a\theta)}=1\cdot 1\cdot\frac{a}{b}\cdot\frac{\lim_{\theta\to 0}\cos(b\theta)}{\lim_{\theta\to 0}\cos(a\theta)}=\frac{a}{b}\cdot\frac{\cos(0)}{\cos(0)}$$
$$=\frac{a}{b}\cdot\frac{1}{1}=\frac{a}{b}.$$

Alternate Solution:

$$\lim_{\theta\to 0}\frac{\tan(a\theta)}{\tan(b\theta)} = \lim_{\theta\to 0}\frac{\tan(a\theta)}{1}\cdot\frac{1}{\tan(b\theta)} = \lim_{\theta\to 0}\frac{\tan(a\theta)}{a\theta}\cdot\frac{b\theta}{\tan(a\theta)}\cdot\frac{a}{b} =$$

$$= \lim_{\theta\to 0}\frac{\tan(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\tan(b\theta)}\cdot\lim_{\theta\to 0}\frac{a}{b} \overset{\#31\ \&\ \#32}{\cong} 1\cdot 1\cdot\frac{a}{b} = \frac{a}{b}.$$

Trig-Identity Heavy Group:

37. $$\lim_{\theta\to 0}\frac{\theta}{\cos\left(\frac{\pi}{2}-\theta\right)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} \overset{\#1}{\cong} 1.$$

Alternative Solution:

$$\lim_{\theta\to 0}\frac{\theta}{\cos\left(\frac{\pi}{2}-\theta\right)} = \lim_{\theta\to 0}\frac{\theta}{\cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)} = \lim_{\theta\to 0}\frac{\theta}{(0)\cos(\theta)+(1)\sin(\theta)} =$$

$$= \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} \overset{\#1}{\cong} 1.$$

38. $$\lim_{\theta\to 0}\frac{\cos\left(\theta+\frac{\pi}{2}\right)}{\theta} = \lim_{\theta\to 0}\frac{\cos(\theta)\cos\left(\frac{\pi}{2}\right)-\sin(\theta)\sin\left(\frac{\pi}{2}\right)}{\theta} = \lim_{\theta\to 0}\frac{\cos(\theta)(0)-\sin(\theta)(1)}{\theta} =$$

$$= \lim_{\theta\to 0}\frac{-\sin(\theta)}{\theta} = -\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = -1.$$

39. $$\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$$

Alternate solution:

$$\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\theta} = \lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)}{\theta} = \lim_{\theta\to 0}\frac{0\cdot\cos(\theta)+1\cdot\sin(\theta)}{\theta} =$$

$$\lim_{\theta\to 0}\frac{0\cdot\cos(\theta)}{\theta}+\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0}0+\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 0+1 = 1.$$

40. $$\lim_{\theta\to 0}\frac{\theta}{\sin\left(\theta+\frac{\pi}{2}\right)} = \lim_{\theta\to 0}\frac{\theta}{\cos(\theta)} \overset{\#16}{\cong} 1.$$

Alternative Solution:

$$\lim_{\theta\to 0}\frac{\theta}{\sin\left(\theta+\frac{\pi}{2}\right)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)\cos\left(\frac{\pi}{2}\right)+\cos(\theta)\sin\left(\frac{\pi}{2}\right)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)(0)+\cos(\theta)(1)} =$$

$$= \lim_{\theta\to 0}\frac{\theta}{0+\cos(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\cos(\theta)} \overset{\#16}{\cong} 1.$$