The Weekly Rigor

No. 38 "A mathematician is a machine for turning coffee into theorems." March 14 2015

101 Problems in Calculating Trigonometric Limits with Solutions
(Part 9)

41. $$\lim_{\theta\to 0}\frac{\sin\left(\frac{\pi}{2}-\theta\right)\tan(\pi+\theta)}{\theta}=\lim_{\theta\to 0}\frac{\cos(\theta)\tan(\theta)}{\theta}=\lim_{\theta\to 0}\cos(\theta)\cdot\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta}=$$
$$=\cos(0)\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta}=1\cdot\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta}=1\cdot 1=1.$$

Alternate solution:

$$\lim_{\theta\to 0}\frac{\sin\left(\frac{\pi}{2}-\theta\right)\tan(\pi+\theta)}{\theta}=\lim_{\theta\to 0}\frac{\left[\sin\left(\frac{\pi}{2}\right)\cos(\theta)-\cos\left(\frac{\pi}{2}\right)\sin(\theta)\right]\left[\frac{\sin(\pi+\theta)}{\cos(\pi+\theta)}\right]}{\theta}=$$
$$=\lim_{\theta\to 0}\frac{[1\cdot\cos(\theta)-0\cdot\sin(\theta)]\left[\frac{\sin(\pi)\cos(\theta)+\cos(\pi)\sin(\theta)}{\cos(\pi)\cos(\theta)-\sin(\pi)\sin(\theta)}\right]}{\theta}=$$
$$=\lim_{\theta\to 0}\frac{\cos(\theta)\cdot\frac{0\cdot\cos(\theta)+(-1)\cdot\sin(\theta)}{(-1)\cdot\cos(\theta)-0\cdot\sin(\theta)}}{\theta}=\lim_{\theta\to 0}\frac{\cos(\theta)\cdot\frac{0-\sin(\theta)}{-\cos(\theta)-0}}{\theta}=$$
$$=\lim_{\theta\to 0}\frac{\cos(\theta)\cdot\frac{-\sin(\theta)}{-\cos(\theta)}}{\theta}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1.$$

42. $$\lim_{\theta\to 0}\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}+\theta\right)=\lim_{\theta\to 0}\cos(\theta)[-\sin(\theta)]=\cos(0)[-\sin(0)]=1\cdot 0=0.$$

Alternate solution:

$$\lim_{\theta\to 0}\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}+\theta\right)=$$
$$=\lim_{\theta\to 0}\left[\left\{\sin\left(\frac{\pi}{2}\right)\cos(\theta)+\cos\left(\frac{\pi}{2}\right)\sin(\theta)\right\}\left\{\cos\left(\frac{\pi}{2}\right)\cos(\theta)-\sin\left(\frac{\pi}{2}\right)\sin(\theta)\right\}\right]=$$
$$=\lim_{\theta\to 0}[\{\cos(\theta)+0\}\{0-\sin(\theta)\}]=\lim_{\theta\to 0}\cos(\theta)[-\sin(\theta)]=\cos(0)[-\sin(0)]=1\cdot 0=0.$$

Let a and b be nonzero numbers.

43. $\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-a\theta\right)}{b\theta}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{b\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{a}{a}=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}=$
$=\frac{a}{b}\cdot 1=\frac{a}{b}.$

Alternate Solution:

$\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-a\theta\right)}{b\theta}=\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}\right)\cos(a\theta)+\sin\left(\frac{\pi}{2}\right)\sin(a\theta)}{b\theta}=$
$=\lim_{\theta\to 0}\frac{(0)\cos(a\theta)+(1)\sin(a\theta)}{b\theta}=\lim_{\theta\to 0}\frac{0+(1)\sin(a\theta)}{b\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}=$
$=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{a}{a}=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}=\frac{a}{b}\cdot 1=\frac{a}{b}.$

Let a and b be nonzero numbers.

44. $\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-a\theta\right)}{\cos\left(\frac{\pi}{2}-b\theta\right)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}\cdot\frac{ab\theta}{ab\theta}=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{b\theta}{\sin(b\theta)}=$
$=\frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)}\overset{\#1}{\cong}\frac{a}{b}\cdot 1\cdot 1=\frac{a}{b}.$

Alternate Solution:

$\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}-a\theta\right)}{\cos\left(\frac{\pi}{2}-b\theta\right)}=\lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}\right)\cos(a\theta)+\sin\left(\frac{\pi}{2}\right)\sin(a\theta)}{\cos\left(\frac{\pi}{2}\right)\cos(b\theta)+\sin\left(\frac{\pi}{2}\right)\sin(b\theta)}=$
$=\lim_{\theta\to 0}\frac{(0)\cos(a\theta)+(1)\sin(a\theta)}{(0)\cos(b\theta)+(1)\sin(b\theta)}=\lim_{\theta\to 0}\frac{0+\sin(a\theta)}{0+\sin(b\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}\cdot\frac{ab\theta}{ab\theta}=$
$=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{b\theta}{\sin(b\theta)}=\frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)}\overset{\#1}{\cong}\frac{a}{b}\cdot 1\cdot 1=\frac{a}{b}.$

Let a and b be nonzero numbers.

45. $\lim_{\theta\to 0}\frac{\sin\left(\frac{\pi}{2}-a\theta\right)}{\sin\left(\frac{\pi}{2}-b\theta\right)}=\lim_{\theta\to 0}\frac{\cos(a\theta)}{\cos(b\theta)}=\frac{\lim_{\theta\to 0}\cos(a\theta)}{\lim_{\theta\to 0}\cos(b\theta)}=\frac{\cos(0)}{\cos(0)}=\frac{1}{1}=1.$

Alternate Solution:

$\lim_{\theta\to 0}\frac{\sin\left(\frac{\pi}{2}-a\theta\right)}{\sin\left(\frac{\pi}{2}-b\theta\right)}=\lim_{\theta\to 0}\frac{\sin\left(\frac{\pi}{2}\right)\cos(a\theta)-\cos\left(\frac{\pi}{2}\right)\sin(a\theta)}{\sin\left(\frac{\pi}{2}\right)\cos(b\theta)-\cos\left(\frac{\pi}{2}\right)\sin(b\theta)}=$
$\lim_{\theta\to 0}\frac{(1)\cos(a\theta)-(0)\sin(a\theta)}{(1)\cos(b\theta)-(0)\sin(b\theta)}=\lim_{\theta\to 0}\frac{\cos(a\theta)}{\cos(b\theta)}=\frac{\lim_{\theta\to 0}\cos(a\theta)}{\lim_{\theta\to 0}\cos(b\theta)}=\frac{\cos(0)}{\cos(0)}=\frac{1}{1}=1.$