The Weekly Rigor

No. 39 “A mathematician is a machine for turning coffee into theorems.” March 21 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 10)

46. $\lim_{\theta\to 0}[\sin(\pi-\theta)+\tan(\pi-\theta)] = \lim_{\theta\to 0}[\sin(\theta)+\tan(\theta)] = \sin(0)+\tan(0) = 0+0 = 0.$

Alternate solution:

$$\lim_{\theta\to 0}[\sin(\pi-\theta)+\tan(\pi-\theta)] = \lim_{\theta\to 0}\left[\sin(\pi)\cos(\theta)-\cos(\pi)\sin(\theta)+\frac{\tan(\pi)+\tan(\theta)}{1-\tan(\pi)\tan(\theta)}\right] =$$

$$\lim_{\theta\to 0}\left[0\cdot\cos(\theta)-(-1)\sin(\theta)+\frac{0+\tan(\theta)}{1-0\cdot\tan(\theta)}\right] = \lim_{\theta\to 0}[\sin(\theta)+\tan(\theta)] =$$

$$= \sin(0)+\tan(0) = 0+0 = 0.$$

Miscellaneous Group:

Let a and b be nonzero numbers.

47. $\lim_{\theta\to 0}\frac{\tan(a\theta)}{\sin(b\theta)} = \lim_{\theta\to 0}\frac{\sin(a\theta)}{\cos(a\theta)}\cdot\frac{1}{\sin(b\theta)} = \lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{1}{\cos(a\theta)}\cdot\frac{1}{\sin(b\theta)} =$

$$= \lim_{\theta\to 0}\frac{\sin(a\theta)}{1}\cdot\frac{a\theta}{a\theta}\frac{1}{\cos(a\theta)}\cdot\frac{1}{\sin(b\theta)}\cdot\frac{b\theta}{b\theta} = \lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{1}{\cos(a\theta)}\cdot\frac{b\theta}{\sin(b\theta)}\cdot\frac{a\theta}{b\theta} =$$

$$= \lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{1}{\cos(a\theta)}\cdot\frac{b\theta}{\sin(b\theta)}\cdot\frac{a}{b} = \frac{a}{b}\cdot\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos(a\theta)}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)} \overset{\#1}{\cong}$$

$$\overset{\#1}{\cong}\frac{a}{b}\cdot 1\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\cos(a\theta)}\cdot 1 = \frac{a}{b}\cdot 1\cdot\frac{1}{\cos(0)}\cdot 1 = \frac{a}{b}\cdot\frac{1}{1} = \frac{a}{b}.$$

48. $\lim_{\theta\to 0}\frac{\sin(\theta)}{1-\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{1}\cdot\frac{1}{1-\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{1}\cdot\frac{1}{1-\cos(\theta)}\cdot\frac{\theta}{\theta} =$

$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\theta}{1-\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{1-\cos(\theta)} = 1\cdot\lim_{\theta\to 0}\frac{\theta}{1-\cos(\theta)} =$$

$$= \lim_{\theta\to 0}\frac{\theta}{1-\cos(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{\theta[1+\cos(\theta)]}{1-\cos^2\theta} = \lim_{\theta\to 0}\frac{\theta[1+\cos(\theta)]}{\sin^2\theta} =$$

$$= \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{1+\cos(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{1+\cos(\theta)}{\sin(\theta)} \overset{\#1}{\cong} 1\cdot\lim_{\theta\to 0}\frac{1+\cos(\theta)}{\sin(\theta)} =$$

$$= \lim_{\theta \to 0} \frac{1+\cos(\theta)}{\sin(\theta)} = \text{Does not exist.}$$

Remark: Note that the numerator is approaching the fixed number 2 while the denominator is approaching 0. Furthermore, $\lim_{\theta \to 0^+} \frac{1+\cos(\theta)}{\sin(\theta)} = +\infty$ while $\lim_{\theta \to 0^-} \frac{1+\cos(\theta)}{\sin(\theta)} = -\infty$.

49. $$\lim_{\theta \to 0} \frac{5\theta + \sin(3\theta)}{\tan(4\theta) - 7\theta\cos(6\theta)} = \lim_{\theta \to 0} \frac{5\theta + \sin(3\theta)}{\frac{\sin(4\theta)}{\cos(4\theta)} - 7\theta\cos(6\theta)} =$$

$$= \lim_{\theta \to 0} \frac{5\theta + \sin(3\theta)}{\frac{\sin(4\theta)}{1} \cdot \frac{1}{\cos(4\theta)} - 7\theta\cos(6\theta)} = \lim_{\theta \to 0} \frac{5\theta + \sin(3\theta)}{\frac{\sin(4\theta)}{1} \cdot \frac{1}{\cos(4\theta)} - 7\theta\cos(6\theta)} \cdot \frac{\left(\frac{1}{\theta}\right)}{\left(\frac{1}{\theta}\right)} =$$

$$= \lim_{\theta \to 0} \frac{\frac{5\theta}{\theta} + \frac{\sin(3\theta)}{\theta}}{\frac{\sin(4\theta)}{\theta} \cdot \frac{1}{\cos(4\theta)} - \frac{7\theta\cos(6\theta)}{\theta}} = \lim_{\theta \to 0} \frac{5 + \frac{\sin(3\theta)}{\theta} \cdot \frac{3}{3}}{\frac{\sin(4\theta)}{\theta} \cdot \frac{4}{4} \cdot \frac{1}{\cos(4\theta)} - \frac{7\cos(6\theta)}{1}} =$$

$$= \lim_{\theta \to 0} \frac{5 + \frac{\sin(3\theta)}{3\theta} \cdot \frac{3}{1}}{\frac{\sin(4\theta)}{4\theta} \cdot \frac{4}{1} \cdot \frac{1}{\cos(4\theta)} - \frac{7\cos(6\theta)}{1}} = \frac{\lim_{\theta \to 0}\left(5 + \frac{\sin(3\theta)}{3\theta} \cdot \frac{3}{1}\right)}{\lim_{\theta \to 0}\left(\frac{\sin(4\theta)}{4\theta} \cdot \frac{4}{1} \cdot \frac{1}{\cos(4\theta)} - \frac{7\cos(6\theta)}{1}\right)} =$$

$$= \frac{\lim_{\theta \to 0} 5 + \lim_{\theta \to 0} \frac{\sin(3\theta)}{3\theta} \cdot \lim_{\theta \to 0} \frac{3}{1}}{\lim_{\theta \to 0} \frac{\sin(4\theta)}{4\theta} \cdot \lim_{\theta \to 0} \frac{4}{1} \cdot \lim_{\theta \to 0} \frac{1}{\cos(4\theta)} - \lim_{\theta \to 0} \frac{7\cos(6\theta)}{1}} =$$

$$= \frac{\lim_{\theta \to 0} 5 + \lim_{\theta \to 0} \frac{\sin(3\theta)}{3\theta} \cdot \lim_{\theta \to 0} \frac{3}{1}}{\lim_{\theta \to 0} \frac{\sin(4\theta)}{4\theta} \cdot \lim_{\theta \to 0} \frac{4}{1} \cdot \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} \cos(4\theta)} - \frac{\lim_{\theta \to 0} 7\cos(6\theta)}{\lim_{\theta \to 0} 1}} = \frac{5 + 1 \cdot 3}{1 \cdot 4 \cdot \frac{1}{\cos(0)} - \frac{7\cos(0)}{1}} =$$

$$= \frac{5 + 1 \cdot 3}{1 \cdot 4 \cdot 1 - 7} = \frac{5+3}{4-7} = \frac{8}{-3}.$$