The Weekly Rigor

No. 40 | "A mathematician is a machine for turning coffee into theorems." | March 28 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 11)

50. $$\lim_{\theta\to 0}\frac{1-\cos^{3}(\theta)}{\sin^{2}(\theta)} \overset{\substack{\text{"difference of}\\ \text{cubes" factoring}\\ \text{formula}}}{=} \lim_{\theta\to 0}\frac{(1-\cos(\theta))\left(1+\cos(\theta)+\cos^{2}(\theta)\right)}{1-\cos^{2}(\theta)} =$$
$$= \lim_{\theta\to 0}\frac{(1-\cos(\theta))\left(1+\cos(\theta)+\cos^{2}(\theta)\right)}{(1+\cos(\theta))(1-\cos(\theta))} = \lim_{\theta\to 0}\frac{1+\cos(\theta)+\cos^{2}(\theta)}{1+\cos(\theta)} =$$
$$= \frac{\lim_{\theta\to 0}(1+\cos(\theta)+\cos^{2}(\theta))}{\lim_{\theta\to 0}(1+\cos(\theta))} = \frac{\lim_{\theta\to 0}1+\lim_{\theta\to 0}\cos(\theta)+\lim_{\theta\to 0}\cos(\theta)\cdot\lim_{\theta\to 0}\cos(\theta)}{\lim_{\theta\to 0}1+\lim_{\theta\to 0}\cos(\theta)} =$$
$$= \frac{1+\cos(0)+\cos(0)\cdot\cos(0)}{1+\cos(0)} = \frac{1+1+1\cdot 1}{1+1} = \frac{3}{2}.$$

51. $$\lim_{\theta\to 0}\frac{2+\sin(\theta)}{3+\theta} = \frac{\lim_{\theta\to 0}[2+\sin(\theta)]}{\lim_{\theta\to 0}[3+\theta]} = \frac{2+\sin(0)}{3+0} = \frac{2+0}{3+0} = \frac{2}{3}.$$

52. $$\lim_{\theta\to 0}\frac{\theta^{2}+1}{\theta+\cos(\theta)} = \frac{\lim_{\theta\to 0}[\theta^{2}+1]}{\lim_{\theta\to 0}[\theta+\cos(\theta)]} = \frac{0^{2}+1}{0+\cos(0)} = \frac{1}{0+1} = 1.$$

53. $$\lim_{\theta\to 0}\frac{\theta+\tan(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\frac{\theta+\frac{\sin(\theta)}{\cos(\theta)}}{\sin(\theta)} = \lim_{\theta\to 0}\frac{\theta+\frac{\sin(\theta)}{\cos(\theta)}}{\frac{\sin(\theta)}{1}} = \lim_{\theta\to 0}\frac{1}{\sin(\theta)}\left(\theta+\frac{\sin(\theta)}{\cos(\theta)}\right) =$$
$$= \lim_{\theta\to 0}\left(\frac{\theta}{\sin(\theta)}+\frac{1}{\cos(\theta)}\right) = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}+\lim_{\theta\to 0}\frac{1}{\cos(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}+\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\cos(\theta)} \overset{\#1}{=}$$
$$\overset{\#1}{=} 1+\frac{1}{\cos(0)} = 1+\frac{1}{1} = 1+1 = 2.$$

54. $$\lim_{\theta\to 0}\frac{1-\cos(3\theta)}{\theta\sin(\theta)}=\lim_{\theta\to 0}\frac{1-\cos(3\theta)}{\theta\sin(\theta)}\cdot\frac{1+\cos(3\theta)}{1+\cos(3\theta)}=\lim_{\theta\to 0}\frac{1-\cos^{2}(3\theta)}{\theta\sin(\theta)\,(1+\cos(3\theta))}=$$

$$=\lim_{\theta\to 0}\frac{\sin^{2}(3\theta)}{\theta\sin(\theta)\,(1+\cos(3\theta))}=\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{\sin(3\theta)}{1}\cdot\frac{1}{\sin(\theta)}\cdot\frac{1}{1+\cos(3\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3}\cdot\frac{\sin(3\theta)}{1}\cdot\frac{3\theta}{3\theta}\cdot\frac{1}{\sin(\theta)}\cdot\frac{1}{1+\cos(3\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{\sin(3\theta)}{3\theta}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{9}{1+\cos(3\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{9}{1+\cos(3\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{\lim_{\theta\to 0}9}{\lim_{\theta\to 0}(1+\cos(3\theta))}\overset{\#1}{\cong}1\cdot 1\cdot 1\cdot\frac{9}{1+\cos(0)}=$$

$$=\frac{9}{1+1}=\frac{9}{2}.$$