The Weekly Rigor

No. 41 "A mathematician is a machine for turning coffee into theorems." April 4 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 12)

55. $$\lim_{\theta\to 0}\frac{1-\cos(2\theta)}{\theta\sin(\theta)}=\lim_{\theta\to 0}\frac{1-\cos(2\theta)}{\theta\sin(\theta)}\cdot\frac{1+\cos(2\theta)}{1+\cos(2\theta)}=\lim_{\theta\to 0}\frac{1-\cos^2(2\theta)}{\theta\sin(\theta)(1+\cos(2\theta))}=$$

$$=\lim_{\theta\to 0}\frac{\sin^2(2\theta)}{\theta\sin(\theta)(1+\cos(2\theta))}=\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\cdot\frac{\sin(2\theta)}{1}\cdot\frac{1}{\sin(\theta)(1+\cos(2\theta))}=$$

$$=\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\cdot\frac{2}{2}\cdot\frac{\sin(2\theta)}{1}\cdot\frac{2\theta}{2\theta}\cdot\frac{1}{\sin(\theta)(1+\cos(2\theta))}=$$

$$=\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\frac{\sin(2\theta)}{2\theta}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{4}{1+\cos(2\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{4}{1+\cos(2\theta)}=$$

$$=\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{\lim_{\theta\to 0}4}{\lim_{\theta\to 0}[1+\cos(2\theta)]}\overset{\#1}{=}1\cdot 1\cdot 1\cdot\frac{4}{1+\cos(0)}=$$

$$=\frac{4}{1+1}=\frac{4}{2}=2.$$

Alternate solution:

$$\lim_{\theta\to 0}\frac{1-\cos(2\theta)}{\theta\sin(\theta)}=\lim_{\theta\to 0}\frac{1-\cos(2\theta)}{\theta\sin(\theta)}\cdot\frac{1+\cos(2\theta)}{1+\cos(2\theta)}=\lim_{\theta\to 0}\frac{1-\cos^2(2\theta)}{\theta\sin(\theta)(1+\cos(2\theta))}=$$

$$=\lim_{\theta\to 0}\frac{\sin^2(2\theta)}{\theta\sin(\theta)(1+\cos(2\theta))}=\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\cdot\frac{\sin(2\theta)}{1}\cdot\frac{1}{\sin(\theta)(1+\cos(2\theta))}=$$

$$=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos(\theta)}{\theta}\cdot\frac{2\sin(\theta)\cos(\theta)}{1}\cdot\frac{1}{\sin(\theta)(1+\cos(2\theta))}=$$

$$=4\lim_{\theta\to 0}\frac{\sin(\theta)\cos(\theta)}{\theta}\cdot\frac{\cos(\theta)}{1}\cdot\frac{1}{(1+\cos(2\theta))}=4\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\cos^2(\theta)}{1+\cos(2\theta)}=$$

$$=4\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\cos^2(\theta)}{1+\cos(2\theta)}=4\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\lim_{\theta\to 0}\cos^2(\theta)}{\lim_{\theta\to 0}[1+\cos(2\theta)]}=4\cdot 1\cdot\frac{\cos^2(0)}{1+\cos(0)}=$$

$$=\frac{4(1)}{1+1}=\frac{4}{2}=2.$$

56. $\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta\sin(\theta)} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta\sin(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\theta\sin(\theta)\,(1+\cos(\theta))} =$

$= \lim_{\theta\to 0}\frac{\sin^2(\theta)}{\theta\sin(\theta)\,(1+\cos(\theta))} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta(1+\cos(\theta))} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{1}{1+\cos(\theta)} =$

$= \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[1+\cos(\theta)]} = 1\cdot\frac{1}{1+\cos(0)} = \frac{1}{1+1} = \frac{1}{2}.$

57. $\lim_{\theta\to 0}\frac{\cos(\theta)}{\theta\csc(\theta)} = \lim_{\theta\to 0}\frac{\cos(\theta)}{1}\cdot\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0}\cos(\theta)\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \cos(0)\cdot 1 = 1\cdot 1 = 1.$

58. $\lim_{\theta\to 0}\sin(2\theta)\cot(\theta) = \lim_{\theta\to 0}\sin(2\theta)\cdot\frac{\cos(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}2\sin(\theta)\cos(\theta)\cdot\frac{\cos(\theta)}{\sin(\theta)} =$

$= \lim_{\theta\to 0}2\cos(\theta)\cos(\theta) = 2\cos(0)\cos(0) = 2\cdot 1\cdot 1 = 2.$

Alternate solution:

$\lim_{\theta\to 0}\sin(2\theta)\cot(\theta) = \lim_{\theta\to 0}\sin(2\theta)\cdot\frac{\cos(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\frac{\sin(2\theta)}{1}\cdot\frac{\cos(\theta)}{1}\cdot\frac{1}{\sin(\theta)} =$

$= \lim_{\theta\to 0}\frac{\sin(2\theta)}{1}\cdot\frac{2\theta}{2\theta}\cdot\frac{\cos(\theta)}{1}\cdot\frac{1}{\sin(\theta)} = \lim_{\theta\to 0}\frac{2}{1}\cdot\frac{\sin(2\theta)}{2\theta}\cdot\frac{\cos(\theta)}{1}\cdot\frac{\theta}{\sin(\theta)} =$

$= \lim_{\theta\to 0}\frac{2}{1}\cdot\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\frac{\cos(\theta)}{1}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} \overset{\#1}{=} 2\cdot 1\cdot\cos(0)\cdot 1 = 2\cdot 1\cdot 1\cdot 1 = 2.$

59. $\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\sin(\theta)(1+\cos(\theta))} =$

$= \lim_{\theta\to 0}\frac{\sin^2(\theta)}{\sin(\theta)(1+\cos(\theta))} = \lim_{\theta\to 0}\frac{\sin(\theta)}{1+\cos(\theta)} = \frac{\lim_{\theta\to 0}\sin(\theta)}{\lim_{\theta\to 0}(1+\cos(\theta))} = \frac{\lim_{\theta\to 0}\sin(\theta)}{\lim_{\theta\to 0}1+\lim_{\theta\to 0}\cos(\theta)} =$

$= \frac{0}{1+\cos(0)} = \frac{0}{1+1} = \frac{0}{2} = 0.$