

The Weekly Rigor

No. 42

“A mathematician is a machine for turning coffee into theorems.”

April 11, 2015

101 Problems in Calculating Trigonometric Limits with Solutions (Part 13)

$$\begin{aligned}
 60. \quad & \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\tan(\theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\tan(\theta)} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2(\theta)}{\tan(\theta)(1 + \cos(\theta))} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{\tan(\theta)(1 + \cos(\theta))} = \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{1} \cdot \frac{1}{\tan(\theta)} \cdot \frac{1}{1 + \cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{1} \cdot \frac{\cot(\theta)}{1} \cdot \frac{1}{1 + \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{1} \cdot \frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{1}{1 + \cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{1} \cdot \frac{\cos(\theta)}{1} \cdot \frac{1}{1 + \cos(\theta)} = \lim_{\theta \rightarrow 0} \sin(\theta) \cdot \lim_{\theta \rightarrow 0} \cos(\theta) \cdot \lim_{\theta \rightarrow 0} \frac{1}{1 + \cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \sin(\theta) \cdot \lim_{\theta \rightarrow 0} \cos(\theta) \cdot \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0} [1 + \cos(\theta)]} = \sin(0) \cdot \cos(0) \cdot \frac{1}{1 + \cos(0)} = \\
 & = 0 \cdot 1 \cdot \frac{1}{1 + 1} = 0 \cdot 1 \cdot \frac{1}{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 61. \quad & \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\tan(\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\left(\frac{\sin(\theta)}{\cos(\theta)} \right)} = \lim_{\theta \rightarrow 0} \frac{\left(\frac{\sin(\theta)}{1} \right)}{\left(\frac{\sin(\theta)}{\cos(\theta)} \right)} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{1} \cdot \frac{\cos(\theta)}{\sin(\theta)} = \lim_{\theta \rightarrow 0} \cos(\theta) = \\
 & = \cos(0) = 1.
 \end{aligned}$$

$$\begin{aligned}
 62. \quad & \lim_{\theta \rightarrow 0} \frac{2 \sin(\theta) - \sin(2\theta)}{\theta \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{2 \sin(\theta) - 2 \sin(\theta) \cos(\theta)}{\theta \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{2 \sin(\theta)(1 - \cos(\theta))}{\theta \cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{2}{1} \cdot \frac{\sin(\theta)}{\theta} \cdot \frac{1 - \cos(\theta)}{\cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{2}{1} \cdot \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\cos(\theta)} = \\
 & = 2 \cdot 1 \cdot \frac{\lim_{\theta \rightarrow 0} [1 - \cos(\theta)]}{\lim_{\theta \rightarrow 0} \cos(\theta)} = 2 \cdot \frac{1 - \cos(0)}{\cos(0)} = 2 \cdot \frac{1 - 1}{1} = 2 \cdot 0 = 0.
 \end{aligned}$$

$$\begin{aligned}
63. \quad & \lim_{\theta \rightarrow 0} \frac{\tan(\theta) - \sin(\theta)}{\theta \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin(\theta)}{\cos(\theta)} - \frac{\sin(\theta)}{1}}{\theta \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{\left(\frac{\sin(\theta) - \cos(\theta) \sin(\theta)}{\cos(\theta)} \right)}{\left(\frac{\theta \cos(\theta)}{1} \right)} = \\
& = \lim_{\theta \rightarrow 0} \frac{\sin(\theta) - \cos(\theta) \sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\theta \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)(1 - \cos(\theta))}{\cos(\theta)} \cdot \frac{1}{\theta \cos(\theta)} = \\
& = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \frac{1 - \cos(\theta)}{\cos^2(\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\cos^2(\theta)} = \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\lim_{\theta \rightarrow 0}(1 - \cos(\theta))}{\lim_{\theta \rightarrow 0} \cos^2(\theta)} = \\
& = 1 \cdot \frac{1 - \cos(0)}{\cos^2(0)} = 1 \cdot \frac{1 - 1}{1} = 1 \cdot 0 = 0.
\end{aligned}$$

$$\begin{aligned}
64. \quad & \lim_{\theta \rightarrow 0} \frac{\csc(\theta) - \cot(\theta)}{\sin(\theta)} = \lim_{\theta \rightarrow 0} \frac{\frac{1}{\sin(\theta)} - \frac{\cos(\theta)}{\sin(\theta)}}{\sin(\theta)} = \lim_{\theta \rightarrow 0} \frac{\left(\frac{1 - \cos(\theta)}{\sin(\theta)} \right)}{\left(\frac{\sin(\theta)}{1} \right)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\sin(\theta)} \cdot \frac{1}{\sin(\theta)} = \\
& = \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\sin^2(\theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\sin^2(\theta)} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2(\theta)}{\sin^2(\theta)(1 + \cos(\theta))} = \\
& = \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{\sin^2(\theta)(1 + \cos(\theta))} = \lim_{\theta \rightarrow 0} \frac{1}{1 + \cos(\theta)} = \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0}[1 + \cos(\theta)]} = \frac{1}{1 + \cos(0)} = \\
& = \frac{1}{1 + 1} = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
65. \quad & \lim_{\theta \rightarrow 0} \frac{2\theta + 1 - \cos(\theta)}{3\theta} = \lim_{\theta \rightarrow 0} \left(\frac{2\theta}{3\theta} + \frac{1 - \cos(\theta)}{3\theta} \right) = \lim_{\theta \rightarrow 0} \frac{2}{3} + \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{3\theta} = \\
& = \frac{2}{3} + \frac{1}{3} \lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\theta} \stackrel{\#19}{=} \frac{2}{3} + \frac{1}{3}(0) = \frac{2}{3}.
\end{aligned}$$

$$\begin{aligned}
66. \quad & \lim_{\theta \rightarrow 0} \frac{\sin^3(\theta)}{(2\theta)^3} = \lim_{\theta \rightarrow 0} \frac{\sin^3(\theta)}{8\theta^3} = \frac{1}{8} \lim_{\theta \rightarrow 0} \frac{\sin^3(\theta)}{\theta^3} = \frac{1}{8} \lim_{\theta \rightarrow 0} \left(\frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\theta} \right) = \\
& = \frac{1}{8} \left(\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \right) = \frac{1}{8} (1 \cdot 1 \cdot 1) = \frac{1}{8}.
\end{aligned}$$

$$\begin{aligned}
67. \quad & \lim_{\theta \rightarrow 0} \frac{4\theta^2 + 3\theta \sin(\theta)}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{\theta(4\theta + 3 \sin(\theta))}{\theta^2} = \lim_{\theta \rightarrow 0} \frac{4\theta + 3 \sin(\theta)}{\theta} = \lim_{\theta \rightarrow 0} \left(\frac{4\theta}{\theta} + \frac{3 \sin(\theta)}{\theta} \right) = \\
& = \lim_{\theta \rightarrow 0} \left(4 + \frac{3 \sin(\theta)}{\theta} \right) = \lim_{\theta \rightarrow 0} 4 + \lim_{\theta \rightarrow 0} \frac{3 \sin(\theta)}{\theta} = 4 + 3 \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 4 + 3(1) = 4 + 3 = 7.
\end{aligned}$$

“Only he who never plays, never loses.”