The Weekly Rigor

No. 42 "A mathematician is a machine for turning coffee into theorems." April 11, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 13)

60. $$\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\tan(\theta)} = \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\tan(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\tan(\theta)(1+\cos(\theta))} =$$
$$= \lim_{\theta\to 0}\frac{\sin^2(\theta)}{\tan(\theta)(1+\cos(\theta))} = \lim_{\theta\to 0}\frac{\sin^2(\theta)}{1}\cdot\frac{1}{\tan(\theta)}\cdot\frac{1}{1+\cos(\theta)} =$$
$$= \lim_{\theta\to 0}\frac{\sin^2(\theta)}{1}\cdot\frac{\cot(\theta)}{1}\cdot\frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0}\frac{\sin^2(\theta)}{1}\cdot\frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{1}{1+\cos(\theta)} =$$
$$= \lim_{\theta\to 0}\frac{\sin(\theta)}{1}\cdot\frac{\cos(\theta)}{1}\cdot\frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0}\sin(\theta)\cdot\lim_{\theta\to 0}\cos(\theta)\cdot\lim_{\theta\to 0}\frac{1}{1+\cos(\theta)} =$$
$$= \lim_{\theta\to 0}\sin(\theta)\cdot\lim_{\theta\to 0}\cos(\theta)\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[1+\cos(\theta)]} = \sin(0)\cdot\cos(0)\cdot\frac{1}{1+\cos(0)} =$$
$$= 0\cdot 1\cdot\frac{1}{1+1} = 0\cdot 1\cdot\frac{1}{2} = 0.$$

61. $$\lim_{\theta\to 0}\frac{\sin(\theta)}{\tan(\theta)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\left(\frac{\sin(\theta)}{\cos(\theta)}\right)} = \lim_{\theta\to 0}\frac{\left(\frac{\sin(\theta)}{1}\right)}{\left(\frac{\sin(\theta)}{\cos(\theta)}\right)} = \lim_{\theta\to 0}\frac{\sin(\theta)}{1}\cdot\frac{\cos(\theta)}{\sin(\theta)} = \lim_{\theta\to 0}\cos(\theta) =$$
$$= \cos(0) = 1.$$

62. $$\lim_{\theta\to 0}\frac{2\sin(\theta)-\sin(2\theta)}{\theta\cos(\theta)} = \lim_{\theta\to 0}\frac{2\sin(\theta)-2\sin(\theta)\cos(\theta)}{\theta\cos(\theta)} = \lim_{\theta\to 0}\frac{2\sin(\theta)(1-\cos(\theta))}{\theta\cos(\theta)} =$$
$$= \lim_{\theta\to 0}\frac{2}{1}\cdot\frac{\sin(\theta)}{\theta}\cdot\frac{1-\cos(\theta)}{\cos(\theta)} = \lim_{\theta\to 0}\frac{2}{1}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\cos(\theta)} =$$
$$= 2\cdot 1\cdot\frac{\lim_{\theta\to 0}[1-\cos(\theta)]}{\lim_{\theta\to 0}\cos(\theta)} = 2\cdot\frac{1-\cos(0)}{\cos(0)} = 2\cdot\frac{1-1}{1} = 2\cdot 0 = 0.$$

63. $$\lim_{\theta\to 0}\frac{\tan(\theta)-\sin(\theta)}{\theta\cos(\theta)}=\lim_{\theta\to 0}\frac{\frac{\sin(\theta)}{\cos(\theta)}-\frac{\sin(\theta)}{1}}{\theta\cos(\theta)}=\lim_{\theta\to 0}\frac{\left(\frac{\sin(\theta)-\cos(\theta)\sin(\theta)}{\cos(\theta)}\right)}{\left(\frac{\theta\cos(\theta)}{1}\right)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(\theta)-\cos(\theta)\sin(\theta)}{\cos(\theta)}\cdot\frac{1}{\theta\cos(\theta)}=\lim_{\theta\to 0}\frac{\sin(\theta)(1-\cos(\theta))}{\cos(\theta)}\cdot\frac{1}{\theta\cos(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{1-\cos(\theta)}{\cos^2(\theta)}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\cos^2(\theta)}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\frac{\lim_{\theta\to 0}(1-\cos(\theta))}{\lim_{\theta\to 0}\cos^2(\theta)}=$$
$$=1\cdot\frac{1-\cos(0)}{\cos^2(0)}=1\cdot\frac{1-1}{1}=1\cdot 0=0.$$

64. $$\lim_{\theta\to 0}\frac{\csc(\theta)-\cot(\theta)}{\sin(\theta)}=\lim_{\theta\to 0}\frac{\frac{1}{\sin(\theta)}-\frac{\cos(\theta)}{\sin(\theta)}}{\frac{\sin(\theta)}{1}}=\lim_{\theta\to 0}\frac{\left(\frac{1-\cos(\theta)}{\sin(\theta)}\right)}{\left(\frac{\sin(\theta)}{1}\right)}=\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin(\theta)}\cdot\frac{1}{\sin(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin^2(\theta)}=\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin^2(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)}=\lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\sin^2(\theta)\,(1+\cos(\theta))}=$$
$$=\lim_{\theta\to 0}\frac{\sin^2(\theta)}{\sin^2(\theta)\,(1+\cos(\theta))}=\lim_{\theta\to 0}\frac{1}{1+\cos(\theta)}=\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}[1+\cos(\theta)]}=\frac{1}{1+\cos(0)}=$$
$$=\frac{1}{1+1}=\frac{1}{2}.$$

65. $$\lim_{\theta\to 0}\frac{2\theta+1-\cos(\theta)}{3\theta}=\lim_{\theta\to 0}\left(\frac{2\theta}{3\theta}+\frac{1-\cos(\theta)}{3\theta}\right)=\lim_{\theta\to 0}\frac{2}{3}+\lim_{\theta\to 0}\frac{1-\cos(\theta)}{3\theta}=$$
$$=\frac{2}{3}+\frac{1}{3}\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}\overset{\#19}{\cong}\frac{2}{3}+\frac{1}{3}(0)=\frac{2}{3}.$$

66. $$\lim_{\theta\to 0}\frac{\sin^3(\theta)}{(2\theta)^3}=\lim_{\theta\to 0}\frac{\sin^3(\theta)}{8\theta^3}=\frac{1}{8}\lim_{\theta\to 0}\frac{\sin^3(\theta)}{\theta^3}=\frac{1}{8}\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta}\right)=$$
$$=\frac{1}{8}\left(\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\right)=\frac{1}{8}(1\cdot 1\cdot 1)=\frac{1}{8}.$$

67. $$\lim_{\theta\to 0}\frac{4\theta^2+3\theta\sin(\theta)}{\theta^2}=\lim_{\theta\to 0}\frac{\theta(4\theta+3\sin(\theta))}{\theta^2}=\lim_{\theta\to 0}\frac{4\theta+3\sin(\theta)}{\theta}=\lim_{\theta\to 0}\left(\frac{4\theta}{\theta}+\frac{3\sin(\theta)}{\theta}\right)=$$
$$=\lim_{\theta\to 0}\left(4+\frac{3\sin(\theta)}{\theta}\right)=\lim_{\theta\to 0}4+\lim_{\theta\to 0}\frac{3\sin(\theta)}{\theta}=4+3\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=4+3(1)=4+3=7.$$