The Weekly Rigor

No. 43 "A mathematician is a machine for turning coffee into theorems." April 18, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 14)

68. $\lim_{\theta\to 0}\frac{\sin[\cos(\theta)]}{\sec(\theta)}=\frac{\lim_{\theta\to 0}\sin[\cos(\theta)]}{\lim_{\theta\to 0}\sec(\theta)}=\frac{\sin\left(\lim_{\theta\to 0}\cos(\theta)\right)}{1}=\frac{\sin(\cos(0))}{1}=\sin(1).$

69. $\lim_{\theta\to 0}\frac{\theta^2}{1-\cos^2(2\theta)}=\lim_{\theta\to 0}\frac{\theta^2}{\sin^2(2\theta)}=\lim_{\theta\to 0}\frac{\theta}{\sin(2\theta)}\cdot\frac{\theta}{\sin(2\theta)}=\lim_{\theta\to 0}\frac{\theta}{\sin(2\theta)}\cdot\frac{2}{2}\cdot\frac{\theta}{\sin(2\theta)}\cdot\frac{2}{2}=$

$=\lim_{\theta\to 0}\frac{2\theta}{\sin(2\theta)}\cdot\frac{2\theta}{\sin(2\theta)}\cdot\frac{1}{4}=\frac{1}{4}\left(\lim_{\theta\to 0}\frac{2\theta}{\sin(2\theta)}\cdot\lim_{\theta\to 0}\frac{2\theta}{\sin(2\theta)}\right)\overset{\#1}{\cong}=\frac{1}{4}(1\cdot 1)=\frac{1}{4}.$

Alternate solution:

$\lim_{\theta\to 0}\frac{\theta^2}{1-\cos^2(2\theta)}=\lim_{\theta\to 0}\frac{\theta^2}{\sin^2(2\theta)}=\lim_{\theta\to 0}\frac{\theta}{\sin(2\theta)}\cdot\frac{\theta}{\sin(2\theta)}=$

$=\lim_{\theta\to 0}\frac{\theta}{2\sin(\theta)\cos(\theta)}\cdot\frac{\theta}{2\sin(\theta)\cos(\theta)}=\lim_{\theta\to 0}\frac{1}{4}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{1}{\cos(\theta)}\cdot\frac{1}{\cos(\theta)}=$

$=\frac{1}{4}\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{1}{\cos(\theta)}\cdot\lim_{\theta\to 0}\frac{1}{\cos(\theta)}\overset{\#1}{\cong}\frac{1}{4}\cdot 1\cdot 1\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\cos(\theta)}\cdot\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\cos(\theta)}=$

$=\frac{1}{4}\cdot\frac{1}{\cos(0)}\cdot\frac{1}{\cos(0)}=\frac{1}{4}\cdot\frac{1}{1}\cdot\frac{1}{1}=\frac{1}{4}.$

70. $\lim_{\theta\to 0}\frac{\sec(6\theta)\tan(3\theta)}{\theta}=\lim_{\theta\to 0}\frac{\tan(3\theta)}{\theta\cos(6\theta)}=\lim_{\theta\to 0}\frac{1}{\cos(6\theta)}\cdot\frac{\sin(3\theta)}{\theta\cos(3\theta)}=$

$=\lim_{\theta\to 0}\frac{1}{\cos(6\theta)}\cdot\frac{\sin(3\theta)}{\theta}\cdot\frac{1}{\cos(3\theta)}=\lim_{\theta\to 0}\frac{1}{\cos(6\theta)}\cdot\frac{\sin(3\theta)}{\theta}\cdot\frac{3}{3}\cdot\frac{1}{\cos(3\theta)}=$

$=3\lim_{\theta\to 0}\frac{1}{\cos(6\theta)}\cdot\frac{\sin(3\theta)}{3\theta}\cdot\frac{1}{\cos(3\theta)}=3\lim_{\theta\to 0}\frac{1}{\cos(6\theta)}\cdot\lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos(3\theta)}=$

$=3\cdot\frac{1}{\cos(0)}\cdot 1\cdot\frac{1}{\cos(0)}=3\cdot 1\cdot 1\cdot 1=3.$

71. $$\lim_{\theta\to 0}\theta^2\cot^2(4\theta)=\lim_{\theta\to 0}\theta^2\cdot\frac{\cos^2(4\theta)}{\sin^2(4\theta)}=\lim_{\theta\to 0}\frac{\theta^2}{\sin^2(4\theta)}\cdot\cos^2(4\theta)=$$
$$=\lim_{\theta\to 0}\frac{\theta}{\sin(4\theta)}\cdot\frac{\theta}{\sin(4\theta)}\cdot\cos(4\theta)\cdot\cos(4\theta)=$$
$$=\lim_{\theta\to 0}\frac{\theta}{\sin(4\theta)}\cdot\frac{4}{4}\cdot\frac{\theta}{\sin(4\theta)}\cdot\frac{4}{4}\cdot\cos(4\theta)\cdot\cos(4\theta)=$$
$$=\frac{1}{16}\lim_{\theta\to 0}\frac{4\theta}{\sin(4\theta)}\cdot\frac{4\theta}{\sin(4\theta)}\cdot\cos(4\theta)\cdot\cos(4\theta)=$$
$$=\frac{1}{16}\lim_{\theta\to 0}\frac{4\theta}{\sin(4\theta)}\cdot\lim_{\theta\to 0}\frac{4\theta}{\sin(4\theta)}\cdot\lim_{\theta\to 0}\cos(4\theta)\cdot\lim_{\theta\to 0}\cos(4\theta)\overset{\#1}{\cong}\frac{1}{16}\cdot 1\cdot 1\cdot\cos(0)\cdot\cos(0)=$$
$$=\frac{1}{16}\cdot 1\cdot 1\cdot 1\cdot 1=\frac{1}{16}.$$

72. $$\lim_{\theta\to 0}\frac{\tan(\pi-\theta)-\theta}{\sin(\theta+\pi)}=\lim_{\theta\to 0}\frac{\frac{\sin(\pi-\theta)}{\cos(\pi-\theta)}-\theta}{\sin(\theta+\pi)}=\lim_{\theta\to 0}\frac{\left[\frac{\sin(\pi)\cos(\theta)-\cos(\pi)\sin(\theta)}{\cos(\pi)\cos(\theta)+\sin(\pi)\sin(\theta)}\right]-\theta}{\sin(\theta)\cos(\pi)+\cos(\theta)\sin(\pi)}=$$
$$=\lim_{\theta\to 0}\frac{\left[\frac{0\cdot\cos(\theta)-(-1)\cdot\sin(\theta)}{(-1)\cdot\cos(\theta)+0\cdot\sin(\theta)}\right]-\theta}{\sin(\theta)\cdot(-1)+\cos(\theta)\cdot 0}=\lim_{\theta\to 0}\frac{\left[\frac{\sin(\theta)}{-\cos(\theta)}\right]-\theta}{-\sin(\theta)}=\lim_{\theta\to 0}\frac{\theta+\frac{\sin(\theta)}{\cos(\theta)}}{\frac{\sin(\theta)}{1}}=$$
$$=\lim_{\theta\to 0}\left[\frac{\theta}{\sin(\theta)}+\frac{1}{\cos(\theta)}\right]=\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}+\lim_{\theta\to 0}\frac{1}{\cos(\theta)}\overset{\#1}{\cong}1+\frac{1}{\cos(0)}=1+\frac{1}{1}=1+1=2.$$

Let a and b be nonzero numbers.

73. $$\lim_{\theta\to 0}\frac{\cos(a\theta)\tan(a\theta)}{b\theta}=\lim_{\theta\to 0}\frac{\cos(a\theta)}{b\theta}\cdot\frac{\sin(a\theta)}{\cos(a\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{b\theta}=\frac{1}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{\theta}\cdot\frac{a}{a}=$$
$$=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}=\frac{a}{b}\cdot 1=\frac{a}{b}.$$

Let a and b be nonzero numbers.

74. $$\lim_{\theta\to 0}\frac{\cos(a\theta)\tan(a\theta)}{\cos(b\theta)\tan(b\theta)}=\lim_{\theta\to 0}\frac{\cos(a\theta)\cdot\left[\frac{\sin(a\theta)}{\cos(a\theta)}\right]}{\cos(b\theta)\cdot\left[\frac{\sin(b\theta)}{\cos(b\theta)}\right]}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}=\lim_{\theta\to 0}\frac{\sin(a\theta)}{\sin(b\theta)}\cdot\frac{ab\theta}{ab\theta}=$$
$$=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\frac{b\theta}{\sin(b\theta)}=\frac{a}{b}\lim_{\theta\to 0}\frac{\sin(a\theta)}{a\theta}\cdot\lim_{\theta\to 0}\frac{b\theta}{\sin(b\theta)}\overset{\#1}{\cong}\frac{a}{b}\cdot 1\cdot 1=\frac{a}{b}.$$