The Weekly Rigor

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101 Problems in Calculating Trigonometric Limits with Solutions

(Part 15)

75. $$\lim_{\theta\to 0}\frac{\sin^2(\theta)+2\cos(\theta)-2}{\cos^2(\theta)-\sin(\theta)-1}=\lim_{\theta\to 0}\frac{1-\cos^2(\theta)+2\cos(\theta)-2}{1-\sin^2(\theta)-\sin(\theta)-1}=$$

$$=\lim_{\theta\to 0}\frac{-[\cos^2(\theta)-2\cos(\theta)+1]}{-[\sin^2(\theta)+\sin(\theta)]}=\lim_{\theta\to 0}\frac{\cos^2(\theta)-2\cos(\theta)+1}{\sin^2(\theta)+\sin(\theta)}=$$

$$=\lim_{\theta\to 0}\frac{(\cos(\theta)-1)(\cos(\theta)-1)}{\sin(\theta)(\sin(\theta)+1)}=\lim_{\theta\to 0}\frac{\cos(\theta)-1}{1}\cdot\frac{\cos(\theta)-1}{1}\cdot\frac{1}{\sin(\theta)}\cdot\frac{1}{\sin(\theta)+1}=$$

$$=\lim_{\theta\to 0}\frac{\cos(\theta)-1}{1}\cdot\frac{\theta}{\theta}\cdot\frac{\cos(\theta)-1}{1}\cdot\frac{\theta}{\theta}\cdot\frac{1}{\sin(\theta)}\cdot\frac{1}{\sin(\theta)+1}=$$

$$=\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)-1}{\theta}\cdot\frac{\theta}{\sin(\theta)}\cdot\frac{\theta}{\sin(\theta)+1}=$$

$$=\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)+1}=$$

$$=0\cdot 0\cdot 1\cdot\frac{\lim_{\theta\to 0}\theta}{\lim_{\theta\to 0}[\sin(\theta)+1]}=0\cdot\frac{0}{0+1}=0.$$

76. $$\lim_{\theta\to 0}\frac{\sin(2\theta)-\tan(2\theta)}{\theta^2}=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos(\theta)-\dfrac{\sin(2\theta)}{\cos(2\theta)}}{\theta^2}=$$

$$\lim_{\theta\to 0}\left[2\sin(\theta)\cos(\theta)-\frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)}\right]\cdot\frac{1}{\theta^2}=$$

$$=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos(\theta)[\cos^2(\theta)-\sin^2(\theta)]-2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)}\cdot\frac{1}{\theta^2}=$$

$$=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos^3(\theta)-2\sin^3(\theta)\cos(\theta)-2\sin(\theta)\cos(\theta)}{[\cos^2(\theta)-\sin^2(\theta)]\theta^2}=$$

$$=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos(\theta)[\cos^2(\theta)-\sin^2(\theta)-1]}{[\cos^2(\theta)-\sin^2(\theta)]\theta^2}=$$

$$= \lim_{\theta\to 0} \frac{2\sin(\theta)\cos(\theta)[1-\sin^2(\theta)-\sin^2(\theta)-1]}{[\cos^2(\theta)-\sin^2(\theta)]\theta^2} = \lim_{\theta\to 0} \frac{2\sin(\theta)\cos(\theta)[-2\sin^2(\theta)]}{[\cos^2(\theta)-\sin^2(\theta)]\theta^2} =$$

$$= -4\lim_{\theta\to 0} \frac{\sin^3(\theta)\cos(\theta)}{[\cos^2(\theta)-\sin^2(\theta)]\theta^2} = -4\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)\cos(\theta)}{[\cos^2(\theta)-\sin^2(\theta)]} =$$

$$= -4\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)\cos(\theta)}{[\cos^2(\theta)-\sin^2(\theta)]} =$$

$$= -4\cdot 1\cdot 1\cdot \frac{\lim_{\theta\to 0}\sin(\theta)\cos(\theta)}{\lim_{\theta\to 0}[\cos^2(\theta)-\sin^2(\theta)]} = -4\cdot\frac{\sin(0)\cos(0)}{\cos^2(0)-\sin^2(0)} = -4\cdot\frac{0\cdot 1}{1-0} = -4\cdot\frac{0}{1} =$$

$$= 0.$$

77. $$\lim_{\theta\to 0}\frac{\sin(\theta)-2\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}-\frac{2\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}-\lim_{\theta\to 0}2 = 1-2 = -1.$$

78. $$\lim_{\theta\to 0}\frac{3-\csc(\theta)}{7-\cot(\theta)} = \lim_{\theta\to 0}\frac{3-\frac{1}{\sin(\theta)}}{7-\frac{\cos(\theta)}{\sin(\theta)}} = \lim_{\theta\to 0}\frac{\frac{3\sin(\theta)-1}{\sin(\theta)}}{\frac{7\sin(\theta)-\cos(\theta)}{\sin(\theta)}} =$$

$$= \lim_{\theta\to 0}\frac{3\sin(\theta)-1}{\sin(\theta)}\cdot\frac{\sin(\theta)}{7\sin(\theta)-\cos(\theta)} = \lim_{\theta\to 0}\frac{3\sin(\theta)-1}{7\sin(\theta)-\cos(\theta)} = \frac{\lim_{\theta\to 0}[3\sin(\theta)-1]}{\lim_{\theta\to 0}[\sin(\theta)-\cos(\theta)]} =$$

$$= \frac{3\sin(0)-1}{\sin(0)-\cos(0)} = \frac{3\cdot 0-1}{0-1} = \frac{-1}{-1} = 1.$$

79. $$\lim_{\theta\to 0}\frac{\theta\cos(\theta)-\sin(\theta)}{\theta} = \lim_{\theta\to 0}\frac{\theta\cos(\theta)}{\theta}-\frac{\sin(\theta)}{\theta} = \lim_{\theta\to 0}\cos(\theta)-\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \cos(0)-1 =$$

$$= 1-1 = 0.$$

80. $$\lim_{\theta\to 0}\frac{\sin(2\theta)\tan(\theta)}{3\theta} = \frac{1}{3}\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\cdot\frac{\tan(\theta)}{1} = \frac{1}{3}\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\cdot\frac{2}{2}\cdot\frac{\tan(\theta)}{1} =$$

$$= \frac{2}{3}\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\frac{\tan(\theta)}{1} = \frac{2}{3}\cdot\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\cdot\lim_{\theta\to 0}\tan(\theta) = \frac{2}{3}\cdot 1\cdot\tan(0) = \frac{2}{3}\cdot 0 = 0.$$