The Weekly Rigor

No. 45 "A mathematician is a machine for turning coffee into theorems." May 2, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 16)

81. $\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)+\tan(\theta)}{3\theta}=\dfrac{1}{3}\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)+\tan(\theta)}{\theta}=\dfrac{1}{3}\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{\theta}+\dfrac{\tan(\theta)}{\theta}=$

$=\dfrac{1}{3}\left[\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{\theta}+\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\right]\overset{\#29}{=}\dfrac{1}{3}\left[\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{\theta}\cdot\dfrac{2}{2}+1\right]=\dfrac{1}{3}\left[2\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{2\theta}+1\right]=$

$=\dfrac{1}{3}[2\cdot 1+1]=\dfrac{1}{3}[2+1]=\dfrac{1}{3}\cdot 3=1.$

82. $\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)-\sin(\theta)}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{\dfrac{\sin(\theta)}{\cos(\theta)}-\dfrac{\sin(\theta)}{1}}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{\dfrac{\sin(\theta)-\sin(\theta)\cos(\theta)}{\cos(\theta)}}{\theta^2}=$

$=\lim\limits_{\theta\to 0}\dfrac{\sin(\theta)[1-\cos(\theta)]}{\theta^2\cos(\theta)}=\lim\limits_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}\cdot\dfrac{1-\cos(\theta)}{\theta}\cdot\dfrac{1}{\cos(\theta)}=$

$=\lim\limits_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1-\cos(\theta)}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1}{\cos(\theta)}\overset{\#1}{=}1\cdot\lim\limits_{\theta\to 0}\dfrac{1-\cos(\theta)}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1}{\cos(\theta)}\overset{\#19}{=}0\cdot\dfrac{1}{\cos(0)}=$

$=0\cdot 1=0.$

83. $\lim\limits_{\theta\to 0}[\csc(\theta)-\cot(\theta)]=\lim\limits_{\theta\to 0}\left[\dfrac{1}{\sin(\theta)}-\dfrac{\cos(\theta)}{\sin(\theta)}\right]=\lim\limits_{\theta\to 0}\dfrac{1-\cos(\theta)}{\sin(\theta)}=$

$=\lim\limits_{\theta\to 0}\dfrac{1-\cos(\theta)}{\sin(\theta)}\cdot\dfrac{1+\cos(\theta)}{1+\cos(\theta)}=\lim\limits_{\theta\to 0}\dfrac{1-\cos^2(\theta)}{\sin(\theta)[1+\cos(\theta)]}=\lim\limits_{\theta\to 0}\dfrac{\sin^2(\theta)}{\sin(\theta)[1+\cos(\theta)]}=$

$=\lim\limits_{\theta\to 0}\dfrac{\sin(\theta)}{1+\cos(\theta)}=\dfrac{\lim\limits_{\theta\to 0}\sin(\theta)}{\lim\limits_{\theta\to 0}[1+\cos(\theta)]}=\dfrac{\sin(0)}{1+\cos(0)}=\dfrac{0}{1+1}=0.$

84. $\lim\limits_{\theta\to 0}\left[\dfrac{1}{\theta^2}-\dfrac{1}{\theta^2\sec(\theta)}\right]=\lim\limits_{\theta\to 0}\dfrac{\sec(\theta)-1}{\theta^2\sec(\theta)}=\lim\limits_{\theta\to 0}\dfrac{\sec(\theta)-1}{\theta^2\sec(\theta)}\cdot\dfrac{\sec(\theta)+1}{\sec(\theta)+1}=$

$=\lim\limits_{\theta\to 0}\dfrac{\sec^2(\theta)-1}{\theta^2\sec(\theta)\,[\sec(\theta)+1]}=\lim\limits_{\theta\to 0}\dfrac{\tan^2(\theta)\cos(\theta)}{\theta^2[\sec(\theta)+1]}=\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\dfrac{\tan(\theta)}{\theta}\cdot\dfrac{\cos(\theta)}{\sec(\theta)+1}=$

$=\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{\cos(\theta)}{\sec(\theta)+1}\overset{\#31}{=}1\cdot 1\cdot\dfrac{\cos(0)}{\sec(0)+1}=\dfrac{1}{1+1}=\dfrac{1}{2}.$

85. $\lim_{\theta\to 0}\dfrac{\sin(\theta)}{\theta+\theta^2}=\lim_{\theta\to 0}\dfrac{\sin(\theta)}{\theta(1+\theta)}=\lim_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}\cdot\dfrac{1}{1+\theta}=\lim_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\dfrac{1}{1+\theta}=1\cdot\dfrac{1}{1+0}=$

$=1\cdot 1=1.$

86. $\lim_{\theta\to 0}\dfrac{\sec(\theta)-1}{\theta^2}=\lim_{\theta\to 0}\dfrac{\sec(\theta)-1}{\theta^2}\cdot\dfrac{\sec(\theta)+1}{\sec(\theta)+1}=\lim_{\theta\to 0}\dfrac{\sec^2(\theta)-1}{\theta^2[\sec(\theta)+1]}=\lim_{\theta\to 0}\dfrac{\tan^2(\theta)}{\theta^2[\sec(\theta)+1]}=$

$=\lim_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\dfrac{\tan(\theta)}{\theta}\cdot\dfrac{1}{\sec(\theta)+1}=\lim_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\lim_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}\cdot\lim_{\theta\to 0}\dfrac{1}{\sec(\theta)+1}\overset{\#31}{=}$

$\overset{\#31}{=}1\cdot 1\cdot\dfrac{1}{1+1}=\dfrac{1}{2}.$

87. $\lim_{\theta\to 0}\dfrac{\cos(2\theta)-\cos(\theta)}{\sin^2(\theta)}=\lim_{\theta\to 0}\dfrac{\cos^2(\theta)-\sin^2(\theta)-\cos(\theta)}{\sin^2(\theta)}=$

$=\lim_{\theta\to 0}\dfrac{\cos^2(\theta)-[1-\cos^2(\theta)]-\cos(\theta)}{\sin^2(\theta)}=\lim_{\theta\to 0}\dfrac{\cos^2(\theta)-1+\cos^2(\theta)-\cos(\theta)}{\sin^2(\theta)}=$

$=\lim_{\theta\to 0}\dfrac{2\cos^2(\theta)-\cos(\theta)-1}{\sin^2(\theta)}=\lim_{\theta\to 0}\dfrac{2\cos^2(\theta)-2\cos(\theta)+\cos(\theta)-1}{\sin^2(\theta)}=$

$=\lim_{\theta\to 0}\dfrac{2\cos(\theta)[\cos(\theta)-1]+\cos(\theta)-1}{\sin^2(\theta)}=\lim_{\theta\to 0}\dfrac{[\cos(\theta)-1][2\cos(\theta)+1]}{\sin^2(\theta)}=$

$=\lim_{\theta\to 0}\dfrac{[\cos(\theta)-1][2\cos(\theta)+1]}{\sin^2(\theta)}=\lim_{\theta\to 0}\dfrac{[\cos(\theta)-1][2\cos(\theta)+1]}{\sin^2(\theta)}\cdot\dfrac{\cos(\theta)+1}{\cos(\theta)+1}=$

$=\lim_{\theta\to 0}\dfrac{[\cos^2(\theta)-1][2\cos(\theta)+1]}{\sin^2(\theta)\,[\cos(\theta)+1]}=\lim_{\theta\to 0}\dfrac{-[1-\cos^2(\theta)][2\cos(\theta)+1]}{\sin^2(\theta)\,[\cos(\theta)+1]}=$

$=\lim_{\theta\to 0}\dfrac{-\sin^2(\theta)[2\cos(\theta)+1]}{\sin^2(\theta)\,[\cos(\theta)+1]}=\lim_{\theta\to 0}\dfrac{-[2\cos(\theta)+1]}{\cos(\theta)+1}=\dfrac{\lim_{\theta\to 0}-[2\cos(\theta)+1]}{\lim_{\theta\to 0}[\cos(\theta)+1]}=$

$=\dfrac{-[2\cos(0)+1]}{\cos(0)+1}=\dfrac{-(2\cdot 1+1)}{1+1}=\dfrac{-3}{2}.$

88. $\lim_{\theta\to 0}\dfrac{\cos(\theta)}{\csc(\theta)}=\lim_{\theta\to 0}\cos(\theta)\sin(\theta)=\lim_{\theta\to 0}\cos(\theta)\cdot\lim_{\theta\to 0}\sin(\theta)=\cos(0)\cdot\sin(0)=1\cdot 0=0.$