The Weekly Rigor

No. 46 “A mathematician is a machine for turning coffee into theorems.” May 9, 2015

101 Problems in Calculating Trigonometric Limits with Solutions

(Part 17)

89. $$\lim_{\theta\to 0}\frac{\theta^3}{\csc(\theta)+1}=\lim_{\theta\to 0}\frac{\theta^3}{\csc(\theta)+1}\cdot\frac{\csc(\theta)-1}{\csc(\theta)-1}=\lim_{\theta\to 0}\frac{\theta^3[\csc(\theta)-1]}{\csc^2(\theta)-1}=\lim_{\theta\to 0}\frac{\theta^3[\csc(\theta)-1]}{\cot^2(\theta)}=$$
$$=\lim_{\theta\to 0}\theta^3\left[\frac{1}{\sin(\theta)}-1\right]\tan^2(\theta)=\lim_{\theta\to 0}\left[\frac{\theta^3}{\sin(\theta)}-\theta^3\right]\frac{\sin^2(\theta)}{\cos^2(\theta)}=$$
$$=\lim_{\theta\to 0}\left[\frac{\theta^3\sin^2(\theta)}{\sin(\theta)\cos^2(\theta)}-\frac{\theta^3\sin^2(\theta)}{\cos^2(\theta)}\right]=\lim_{\theta\to 0}\frac{\theta^3\sin^2(\theta)}{\sin(\theta)\cos^2(\theta)}-\lim_{\theta\to 0}\frac{\theta^3\sin^2(\theta)}{\cos^2(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\theta^3\sin(\theta)}{\cos^2(\theta)}-\lim_{\theta\to 0}\frac{\theta^3\sin^2(\theta)}{\cos^2(\theta)}=\frac{0^3\sin(0)}{\cos^2(0)}-\frac{0^3\sin^2(0)}{\cos^2(0)}=\frac{0\cdot 0}{1}-\frac{0\cdot 0}{1}=0-0=0.$$

90. $$\lim_{\theta\to 0}2\theta^2\sec^2(\theta)\cot^2(\theta)=2\lim_{\theta\to 0}\frac{\theta^2}{\cos^2(\theta)}\cdot\frac{\cos^2(\theta)}{\sin^2(\theta)}=2\lim_{\theta\to 0}\frac{\theta^2}{\sin^2(\theta)}\overset{\#1}{=}2\cdot 1=2.$$

91. $$\lim_{\theta\to 0}\frac{\cot^4(\theta)\tan(\theta)+\sin^2(\theta)-\csc(\theta)+\sec(\theta)=}{\theta^{-3}}$$
$$\lim_{\theta\to 0}\theta^3[\cot^4(\theta)\tan(\theta)+\sin^2(\theta)-\csc(\theta)+\sec(\theta)]=$$
$$=\lim_{\theta\to 0}\theta^3\left[\frac{\cos^4(\theta)}{\sin^4(\theta)}\cdot\frac{\sin(\theta)}{\cos(\theta)}+\sin^2(\theta)-\frac{1}{\sin(\theta)}+\frac{1}{\cos(\theta)}\right]=$$
$$=\lim_{\theta\to 0}\theta^3\left[\frac{\cos^3(\theta)}{\sin^3(\theta)}+\sin^2(\theta)-\frac{1}{\sin(\theta)}+\frac{1}{\cos(\theta)}\right]=$$
$$=\lim_{\theta\to 0}\frac{\theta^3\cos^3(\theta)}{\sin^3(\theta)}+\lim_{\theta\to 0}\theta^3\sin^2(\theta)-\lim_{\theta\to 0}\frac{\theta^3}{\sin(\theta)}+\lim_{\theta\to 0}\frac{\theta^3}{\cos(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\theta^3}{\sin^3(\theta)}\cdot\lim_{\theta\to 0}\frac{\cos^3(\theta)}{1}+\lim_{\theta\to 0}\theta^3\cdot\lim_{\theta\to 0}\sin^2(\theta)-\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\theta^2+\lim_{\theta\to 0}\frac{\theta^3}{\cos(\theta)}=$$
$$=\lim_{\theta\to 0}\frac{\theta^3}{\sin^3(\theta)}\cdot\frac{\cos^3(0)}{1}+0^3\cdot\sin^2(0)-\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot 0^2+\frac{0^3}{\cos(0)}\overset{\#1}{=}1\cdot\frac{1}{1}+0-1\cdot 0+\frac{0}{1}=$$
$$=1+0-0+0=1.$$

92. $\lim_{\theta\to 0}\left[3\sec(\theta)-\frac{\theta^3\csc^3(\theta)}{\cos^3(\theta)}+\theta^2\csc(\theta)\right]=\lim_{\theta\to 0}\left[\frac{3}{\cos(\theta)}-\frac{\theta^3}{\sin^3(\theta)\cos^3(\theta)}+\frac{\theta^2}{\sin(\theta)}\right]=$

$=\lim_{\theta\to 0}\frac{3}{\cos(\theta)}-\lim_{\theta\to 0}\frac{\theta^3}{\sin^3(\theta)\cos^3(\theta)}+\lim_{\theta\to 0}\frac{\theta^2}{\sin(\theta)}=$

$=\lim_{\theta\to 0}\frac{3}{\cos(\theta)}-\lim_{\theta\to 0}\frac{\theta^3}{\sin^3(\theta)}\cdot\lim_{\theta\to 0}\frac{1}{\cos^3(\theta)}+\lim_{\theta\to 0}\frac{\theta^2}{\sin(\theta)}=$

$=\frac{3}{\cos(0)}-\lim_{\theta\to 0}\frac{\theta^3}{\sin^3(\theta)}\cdot\frac{1}{\cos^3(0)}+\lim_{\theta\to 0}\frac{\theta^2}{\sin(\theta)}\overset{\#1}{\cong}\frac{3}{1}-1\cdot\frac{1}{1}\cdot\frac{1}{1}+1=3-1+1=3.$

93. $\lim_{\theta\to 0}\left[\cos(\theta)-\sin^3(\theta)\csc^2(\theta)-\tan(\theta)\right]=\lim_{\theta\to 0}\left[\cos(\theta)-\frac{\sin^3(\theta)}{\sin^2(\theta)}-\tan(\theta)\right]=$

$\lim_{\theta\to 0}\cos(\theta)-\lim_{\theta\to 0}\sin(\theta)-\lim_{\theta\to 0}\tan(\theta)=\cos(0)-\sin(0)-\tan(0)=1-0-0=1.$

94. $\lim_{\theta\to 0}[8\theta^2\csc^2(\theta)+\tan(\theta)\cos(\theta)]=\lim_{\theta\to 0}\left[\frac{8\theta^2}{\sin^2(\theta)}+\sin(\theta)\right]=$

$=8\lim_{\theta\to 0}\frac{\theta^2}{\sin^2(\theta)}+\lim_{\theta\to 0}\sin(\theta)\overset{\#1}{\cong}8\cdot 1+\sin(0)=8+0=8.$

95. $\lim_{\theta\to 0}2\theta\cot(\theta)\sec(\theta)=2\lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\frac{\cos(\theta)}{1}\cdot\frac{1}{\cos(\theta)}\overset{\#1}{\cong}2\cdot 1\cdot\frac{\cos(0)}{1}\cdot\frac{1}{\cos(0)}=$

$=2\cdot 1\cdot\frac{1}{1}\cdot\frac{1}{1}=2.$

96. $\lim_{\theta\to 0}\left[\frac{\cot(\theta)}{\csc(\theta)}+\sec(\theta)\right]=\lim_{\theta\to 0}\left[\frac{\sin(\theta)}{1}\cdot\frac{\cos(\theta)}{\sin(\theta)}+\frac{1}{\cos(\theta)}\right]=\lim_{\theta\to 0}\left[\cos(\theta)+\frac{1}{\cos(\theta)}\right]=$

$=1+1=2.$

97. $\lim_{\theta\to 0}\frac{\sin(2\theta)\cos^3(\theta)}{\sin(\theta)}=\lim_{\theta\to 0}\frac{2\sin(\theta)\cos(\theta)\cos^3(\theta)}{\sin(\theta)}=2\lim_{\theta\to 0}\cos^4(\theta)=2\cos^4(0)=$

$=2\cdot 1=2.$