

The Weekly Rigor

No. 46

“A mathematician is a machine for turning coffee into theorems.”

May 9, 2015

101 Problems in Calculating Trigonometric Limits with Solutions (Part 17)

$$\begin{aligned}
 89. \quad & \lim_{\theta \rightarrow 0} \frac{\theta^3}{\csc(\theta) + 1} = \lim_{\theta \rightarrow 0} \frac{\theta^3}{\csc(\theta) + 1} \cdot \frac{\csc(\theta) - 1}{\csc(\theta) - 1} = \lim_{\theta \rightarrow 0} \frac{\theta^3[\csc(\theta) - 1]}{\csc^2(\theta) - 1} = \lim_{\theta \rightarrow 0} \frac{\theta^3[\csc(\theta) - 1]}{\cot^2(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \theta^3 \left[\frac{1}{\sin(\theta)} - 1 \right] \tan^2(\theta) = \lim_{\theta \rightarrow 0} \left[\frac{\theta^3}{\sin(\theta)} - \theta^3 \right] \frac{\sin^2(\theta)}{\cos^2(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \left[\frac{\theta^3 \sin^2(\theta)}{\sin(\theta) \cos^2(\theta)} - \frac{\theta^3 \sin^2(\theta)}{\cos^2(\theta)} \right] = \lim_{\theta \rightarrow 0} \frac{\theta^3 \sin^2(\theta)}{\sin(\theta) \cos^2(\theta)} - \lim_{\theta \rightarrow 0} \frac{\theta^3 \sin^2(\theta)}{\cos^2(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\theta^3 \sin(\theta)}{\cos^2(\theta)} - \lim_{\theta \rightarrow 0} \frac{\theta^3 \sin^2(\theta)}{\cos^2(\theta)} = \frac{0^3 \sin(0)}{\cos^2(0)} - \frac{0^3 \sin^2(0)}{\cos^2(0)} = \frac{0 \cdot 0}{1} - \frac{0 \cdot 0}{1} = 0 - 0 = 0.
 \end{aligned}$$

$$90. \quad \lim_{\theta \rightarrow 0} 2\theta^2 \sec^2(\theta) \cot^2(\theta) = 2 \lim_{\theta \rightarrow 0} \frac{\theta^2}{\cos^2(\theta)} \cdot \frac{\cos^2(\theta)}{\sin^2(\theta)} = 2 \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin^2(\theta)} \stackrel{\#1}{=} 2 \cdot 1 = 2.$$

$$\begin{aligned}
 91. \quad & \lim_{\theta \rightarrow 0} \frac{\cot^4(\theta) \tan(\theta) + \sin^2(\theta) - \csc(\theta) + \sec(\theta)}{\theta^{-3}} = \\
 & \lim_{\theta \rightarrow 0} \theta^3 [\cot^4(\theta) \tan(\theta) + \sin^2(\theta) - \csc(\theta) + \sec(\theta)] = \\
 & = \lim_{\theta \rightarrow 0} \theta^3 \left[\frac{\cos^4(\theta)}{\sin^4(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)} + \sin^2(\theta) - \frac{1}{\sin(\theta)} + \frac{1}{\cos(\theta)} \right] = \\
 & = \lim_{\theta \rightarrow 0} \theta^3 \left[\frac{\cos^3(\theta)}{\sin^3(\theta)} + \sin^2(\theta) - \frac{1}{\sin(\theta)} + \frac{1}{\cos(\theta)} \right] = \\
 & = \lim_{\theta \rightarrow 0} \frac{\theta^3 \cos^3(\theta)}{\sin^3(\theta)} + \lim_{\theta \rightarrow 0} \theta^3 \sin^2(\theta) - \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin(\theta)} + \lim_{\theta \rightarrow 0} \frac{\theta^3}{\cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin^3(\theta)} \cdot \lim_{\theta \rightarrow 0} \frac{\cos^3(\theta)}{1} + \lim_{\theta \rightarrow 0} \theta^3 \cdot \lim_{\theta \rightarrow 0} \sin^2(\theta) - \lim_{\theta \rightarrow 0} \frac{\theta}{\sin(\theta)} \cdot \lim_{\theta \rightarrow 0} \theta^2 + \lim_{\theta \rightarrow 0} \frac{\theta^3}{\cos(\theta)} = \\
 & = \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin^3(\theta)} \cdot \frac{\cos^3(0)}{1} + 0^3 \cdot \sin^2(0) - \lim_{\theta \rightarrow 0} \frac{\theta}{\sin(\theta)} \cdot 0^2 + \frac{0^3}{\cos(0)} \stackrel{\#1}{=} 1 \cdot \frac{1}{1} + 0 - 1 \cdot 0 + \frac{0}{1} = \\
 & = 1 + 0 - 0 + 0 = 1.
 \end{aligned}$$

$$\begin{aligned}
92. \quad & \lim_{\theta \rightarrow 0} \left[3 \sec(\theta) - \frac{\theta^3 \csc^3(\theta)}{\cos^3(\theta)} + \theta^2 \csc(\theta) \right] = \lim_{\theta \rightarrow 0} \left[\frac{3}{\cos(\theta)} - \frac{\theta^3}{\sin^3(\theta) \cos^3(\theta)} + \frac{\theta^2}{\sin(\theta)} \right] = \\
& = \lim_{\theta \rightarrow 0} \frac{3}{\cos(\theta)} - \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin^3(\theta) \cos^3(\theta)} + \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin(\theta)} = \\
& = \lim_{\theta \rightarrow 0} \frac{3}{\cos(\theta)} - \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin^3(\theta)} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos^3(\theta)} + \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin(\theta)} = \\
& = \frac{3}{\cos(0)} - \lim_{\theta \rightarrow 0} \frac{\theta^3}{\sin^3(\theta)} \cdot \frac{1}{\cos^3(0)} + \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin(\theta)} \stackrel{\#1}{=} \frac{3}{1} - 1 \cdot \frac{1}{1} \cdot \frac{1}{1} + 1 = 3 - 1 + 1 = 3.
\end{aligned}$$

$$\begin{aligned}
93. \quad & \lim_{\theta \rightarrow 0} [\cos(\theta) - \sin^3(\theta) \csc^2(\theta) - \tan(\theta)] = \lim_{\theta \rightarrow 0} \left[\cos(\theta) - \frac{\sin^3(\theta)}{\sin^2(\theta)} - \tan(\theta) \right] = \\
& \lim_{\theta \rightarrow 0} \cos(\theta) - \lim_{\theta \rightarrow 0} \sin(\theta) - \lim_{\theta \rightarrow 0} \tan(\theta) = \cos(0) - \sin(0) - \tan(0) = 1 - 0 - 0 = 1.
\end{aligned}$$

$$\begin{aligned}
94. \quad & \lim_{\theta \rightarrow 0} [8\theta^2 \csc^2(\theta) + \tan(\theta) \cos(\theta)] = \lim_{\theta \rightarrow 0} \left[\frac{8\theta^2}{\sin^2(\theta)} + \sin(\theta) \right] = \\
& = 8 \lim_{\theta \rightarrow 0} \frac{\theta^2}{\sin^2(\theta)} + \lim_{\theta \rightarrow 0} \sin(\theta) \stackrel{\#1}{=} 8 \cdot 1 + \sin(0) = 8 + 0 = 8.
\end{aligned}$$

$$\begin{aligned}
95. \quad & \lim_{\theta \rightarrow 0} 2\theta \cot(\theta) \sec(\theta) = 2 \lim_{\theta \rightarrow 0} \frac{\theta}{\sin(\theta)} \cdot \frac{\cos(\theta)}{1} \cdot \frac{1}{\cos(\theta)} \stackrel{\#1}{=} 2 \cdot 1 \cdot \frac{\cos(0)}{1} \cdot \frac{1}{\cos(0)} = \\
& = 2 \cdot 1 \cdot \frac{1}{1} \cdot \frac{1}{1} = 2.
\end{aligned}$$

$$\begin{aligned}
96. \quad & \lim_{\theta \rightarrow 0} \left[\frac{\cot(\theta)}{\csc(\theta)} + \sec(\theta) \right] = \lim_{\theta \rightarrow 0} \left[\frac{\sin(\theta)}{1} \cdot \frac{\cos(\theta)}{\sin(\theta)} + \frac{1}{\cos(\theta)} \right] = \lim_{\theta \rightarrow 0} \left[\cos(\theta) + \frac{1}{\cos(\theta)} \right] = \\
& = 1 + 1 = 2.
\end{aligned}$$

$$\begin{aligned}
97. \quad & \lim_{\theta \rightarrow 0} \frac{\sin(2\theta) \cos^3(\theta)}{\sin(\theta)} = \lim_{\theta \rightarrow 0} \frac{2 \sin(\theta) \cos(\theta) \cos^3(\theta)}{\sin(\theta)} = 2 \lim_{\theta \rightarrow 0} \cos^4(\theta) = 2 \cos^4(0) = \\
& = 2 \cdot 1 = 2.
\end{aligned}$$

“Only he who never plays, never loses.”