The Weekly Rigor

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The Interrelationship of the Six Trigonometric Functions via the Reciprocal and Pythagorean Identities

The purpose of this article is to justify the entries in the following chart by means of the Reciprocal and Pythagorean Identities.

Each trigonometric function in terms of the other five

in terms of	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$\sin(\theta)$	$\sin(\theta)$	$\pm\sqrt{1-\cos^2(\theta)}$	$\frac{\pm\tan(\theta)}{\sqrt{\tan^2(\theta)+1}}$	$\frac{1}{\csc(\theta)}$	$\frac{\pm\sqrt{\sec^2(\theta)-1}}{\sec(\theta)}$	$\frac{\pm 1}{\sqrt{\cot^2(\theta)+1}}$
$\cos(\theta)$	$\pm\sqrt{1-\sin^2(\theta)}$	$\cos(\theta)$	$\frac{\pm 1}{\sqrt{\tan^2(\theta)+1}}$	$\frac{\pm\sqrt{\csc^2(\theta)-1}}{\csc(\theta)}$	$\frac{1}{\sec(\theta)}$	$\frac{\pm\cot(\theta)}{\sqrt{\cot^2(\theta)+1}}$
$\tan(\theta)$	$\frac{\pm\sin(\theta)}{\sqrt{1-\sin^2(\theta)}}$	$\frac{\pm\sqrt{1-\cos^2(\theta)}}{\cos(\theta)}$	$\tan(\theta)$	$\frac{\pm 1}{\sqrt{\csc^2(\theta)-1}}$	$\pm\sqrt{\sec^2(\theta)-1}$	$\frac{1}{\cot(\theta)}$
$\csc(\theta)$	$\frac{1}{\sin(\theta)}$	$\frac{\pm 1}{\sqrt{1-\cos^2(\theta)}}$	$\frac{\pm\sqrt{\tan^2(\theta)+1}}{\tan(\theta)}$	$\csc(\theta)$	$\frac{\pm\sec(\theta)}{\sqrt{\sec^2(\theta)-1}}$	$\pm\sqrt{\cot^2(\theta)+1}$
$\sec(\theta)$	$\frac{\pm 1}{\sqrt{1-\sin^2(\theta)}}$	$\frac{1}{\cos(\theta)}$	$\pm\sqrt{\tan^2(\theta)+1}$	$\frac{\pm\csc(\theta)}{\sqrt{\csc^2(\theta)-1}}$	$\sec(\theta)$	$\frac{\pm\sqrt{\cot^2(\theta)+1}}{\cot(\theta)}$
$\cot(\theta)$	$\frac{\pm\sqrt{1-\sin^2(\theta)}}{\sin(\theta)}$	$\frac{\pm\cos(\theta)}{\sqrt{1-\cos^2(\theta)}}$	$\frac{1}{\tan(\theta)}$	$\pm\sqrt{\csc^2(\theta)-1}$	$\frac{\pm 1}{\sqrt{\sec^2(\theta)-1}}$	$\cot(\theta)$

The Reciprocal Identities are the following:

$\sec(\theta) = \frac{1}{\cos(\theta)}$ $\csc(\theta) = \frac{1}{\sin(\theta)}$ $\cot(\theta) = \frac{1}{\tan(\theta)}$

The Pythagorean Identities are the following:

$\sin^2(\theta) + \cos^2(\theta) = 1$ $\tan^2(\theta) + 1 = \sec^2(\theta)$ $\cot^2(\theta) + 1 = \csc^2(\theta)$

From the Pythagorean Identities, the following useful identities can be derived:

$$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)} \qquad \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$$

$$\tan(\theta) = \pm\sqrt{\sec^2(\theta)-1} \qquad \sec(\theta) = \pm\sqrt{\tan^2(\theta)+1}$$

$$\cot(\theta) = \pm\sqrt{\csc^2(\theta)-1} \qquad \csc(\theta) = \pm\sqrt{\cot^2(\theta)+1}$$

Now we are ready to derive the values in the above chart. In the following six equations, the “R” and “P” above the equals sign specify that either the Reciprocal or Pythagorean Identities justify the equality.

$$\sin(\theta) \overset{P}{=} \pm\sqrt{1-\cos^2(\theta)} \overset{R}{=} \frac{\pm\sqrt{\sec^2(\theta)-1}}{\sec(\theta)} \overset{P}{=} \frac{\pm\tan(\theta)}{\sqrt{\tan^2(\theta)+1}} \overset{R}{=} \frac{\frac{\pm 1}{\cot(\theta)}}{\sqrt{\frac{1+\cot^2(\theta)}{\cot^2(\theta)}}} = \frac{\pm 1}{\sqrt{1+\cot^2(\theta)}} \overset{P}{=} \frac{1}{\csc(\theta)} \overset{R}{=} \sin(\theta)$$

$$\cos(\theta) \overset{P}{=} \pm\sqrt{1-\sin^2(\theta)} \overset{R}{=} \frac{\pm\sqrt{\csc^2(\theta)-1}}{\csc(\theta)} \overset{P}{=} \frac{\pm\cot(\theta)}{\sqrt{\cot^2(\theta)+1}} \overset{R}{=} \frac{\frac{\pm 1}{\tan(\theta)}}{\sqrt{\frac{1+\tan^2(\theta)}{\tan^2(\theta)}}} = \frac{\pm 1}{\sqrt{1+\tan^2(\theta)}} \overset{P}{=} \frac{1}{\sec(\theta)} \overset{R}{=} \cos(\theta)$$

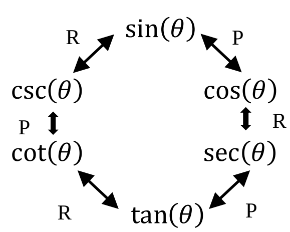
$$\tan(\theta) \overset{P}{=} \pm\sqrt{\sec^2(\theta)-1} \overset{R}{=} \frac{\pm\sqrt{1-\cos^2(\theta)}}{\cos(\theta)} \overset{P}{=} \frac{\pm\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} \overset{R}{=} \frac{\frac{\pm 1}{\csc(\theta)}}{\sqrt{\frac{\csc^2(\theta)-1}{\csc^2(\theta)}}} = \frac{\pm 1}{\sqrt{\csc^2(\theta)-1}} \overset{P}{=} \frac{1}{\cot(\theta)} \overset{R}{=} \tan(\theta)$$

$$\csc(\theta) \overset{P}{=} \pm\sqrt{\cot^2(\theta)+1} \overset{R}{=} \frac{\pm\sqrt{1+\tan^2(\theta)}}{\tan(\theta)} \overset{P}{=} \frac{\pm\sec(\theta)}{\sqrt{\sec^2(\theta)-1}} \overset{R}{=} \frac{\frac{\pm 1}{\cos(\theta)}}{\sqrt{\frac{1-\cos^2(\theta)}{\cos^2(\theta)}}} = \frac{\pm 1}{\sqrt{1-\cos^2(\theta)}} \overset{P}{=} \frac{1}{\sin(\theta)} \overset{R}{=} \csc(\theta)$$

$$\sec(\theta) \overset{P}{=} \pm\sqrt{\tan^2(\theta)+1} \overset{R}{=} \frac{\pm\sqrt{1+\cot^2(\theta)}}{\cot(\theta)} \overset{P}{=} \frac{\pm\csc(\theta)}{\sqrt{\csc^2(\theta)-1}} \overset{R}{=} \frac{\frac{\pm 1}{\sin(\theta)}}{\sqrt{\frac{1-\sin^2(\theta)}{\sin^2(\theta)}}} = \frac{\pm 1}{\sqrt{1-\sin^2(\theta)}} \overset{P}{=} \frac{1}{\cos(\theta)} \overset{R}{=} \sec(\theta)$$

$$\cot(\theta) \overset{P}{=} \pm\sqrt{\csc^2(\theta)-1} \overset{R}{=} \frac{\pm\sqrt{1-\sin^2(\theta)}}{\sin(\theta)} \overset{P}{=} \frac{\pm\cos(\theta)}{\sqrt{1-\cos^2(\theta)}} \overset{R}{=} \frac{\frac{\pm 1}{\sec(\theta)}}{\sqrt{\frac{\sec^2(\theta)-1}{\sec^2(\theta)}}} = \frac{\pm 1}{\sqrt{\sec^2(\theta)-1}} \overset{P}{=} \frac{1}{\tan(\theta)} \overset{R}{=} \cot(\theta)$$

The above six equations indicate a sequence of transitions from one trigonometric function to another:



In Memoriam
John Forbes Nash, Jr.
1928-2015

“Only he who never plays, never loses.”