The Weekly Rigor

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Generating Pythagorean Triples Using a Series of Consecutive Odd Positive Integers

INTRODUCTION

Consider the following list of series consisting of consecutive odd positive integers:

$1 = 1$
$1 + 3 = 4$
$1 + 3 + 5 = 9$
$1 + 3 + 5 + 7 = 16$
$1 + 3 + 5 + 7 + 9 = 25$
$1 + 3 + 5 + 7 + 9 + 11 = 36$
$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 = 121$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 = 144$
$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 = 169$

The first evident pattern concerns the sum of each series, which equals the square of the number of terms in the series. We will prove the pattern in general in Theorem 1. In Theorem 2, we will prove a less obvious pattern concerning the last term in the series: If the last term is a perfect square, then that term, the sum of all the previous terms, and the final sum of the entire series is a Pythagorean triple. For example, in the series $1 + 3 + 5 + 7 + 9 = 25$, 9 is a perfect square, the previous terms add up to 16, another perfect square, and the sum of the entire series is also a perfect square.

Definition 1: A *perfect square* is an integer that is the square of an integer.

Definition 2: A *Pythagorean triple* is a set of three positive integers *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$.

Theorem 1: $\sum_{i=1}^{n}(2i-1)=n^2$.

Proof: By induction on $n \in \mathbb{Z}^+$.

I. $\sum_{i=1}^{1}(2i-1)=2(1)-1=1=1^2$.

II. Suppose for $k \in \mathbb{Z}^+$ that $\sum_{i=1}^{k}(2i-1)=k^2$. Hence, $\sum_{i=1}^{k+1}(2i-1)=$
$=\sum_{i=1}^{k}(2i-1)+\sum_{i=k+1}^{k+1}(2i-1)\overset{IH}{=}k^2+2(k+1)-1=k^2+2k+1=(k+1)^2$.
Consequently, if $\sum_{i=1}^{k}(2i-1)=k^2$, then $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$.

Therefore, the theorem holds, by mathematical induction.

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Theorem 2: If the *n*th term of $\sum_{i=1}^{n}(2i-1)$ is a perfect square, then *n* – 1, *k*, and *n* is a Pythagorean triple, for some integer *k*.

Proof: Suppose that the *n*th term of $\sum_{i=1}^{n}(2i-1)$ is a perfect square. Hence, $2n-1=k^2$ for some integer *k*, by Definition 1. $\sum_{i=1}^{n}(2i-1)=n^2$ and $\sum_{i=1}^{n-1}(2i-1)=(n-1)^2$, by Theorem 1. So, $\sum_{i=1}^{n}(2i-1)=\sum_{i=1}^{n-1}(2i-1)+(2n-1)=(n-1)^2+k^2=n^2$. Therefore, *n* – 1, *k*, and *n* is a Pythagorean triple, for some integer *k*, by Definition 2.

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