

The Weekly Rigor

51 Problems in Calculating Integrals Using U -Substitution with Solutions (Part 1)

INTRODUCTION

There are up to nine integration techniques that the calculus student may need to learn: u -substitution, integration by parts, trigonometric integrals, trigonometric substitution, quadratic polynomials, partial fractions, rationalizing substitution, numerical integration, and improper integrals. Usually these techniques are presented in this order, but there are exceptions.

In working with students, I found it extremely helpful to separate u -substitution problems into different categories. It turns out that there are four obvious types, distinguishable on the basis of how the “ u -assignment” work plays out. An example of Type 1 is

$$\int (x + 1)^4 dx,$$

where the u -assignment work looks like

$$\begin{aligned} u &= x + 1 \\ du &= dx. \end{aligned}$$

An example of Type 2 is

$$\int (3x + 1)^4 dx,$$

where the u -assignment work looks like

$$\begin{aligned} u &= 3x + 1 \\ du &= 3dx \\ \frac{du}{3} &= dx. \end{aligned}$$

An example of Type 3 is

$$\int (3x^2 + 1)^4 x dx,$$

where the u -assignment work looks like

$$\begin{aligned} u &= 3x^2 + 1 \\ du &= 6x dx \\ \frac{du}{6} &= x dx. \end{aligned}$$

Type 4 problems are distinctive due to an extra “backward substitution” step. An example of such a problem is

$$\int (x + 3)(x - 1)^4 dx,$$

where the u -assignment work looks like

$$\begin{array}{l} u = x - 1 \\ du = dx \\ u + 1 = x \\ u + 4 = x + 3. \end{array} \quad \left. \vphantom{\begin{array}{l} u = x - 1 \\ du = dx \\ u + 1 = x \\ u + 4 = x + 3. \end{array}} \right\} \begin{array}{|l} \hline \text{“backward} \\ \text{substitution”} \\ \hline \end{array}$$

The main body of problems in this article are divided into the four types.

An additional aid to the new student of u -substitution is the identification of the four-step procedure in Types 1-3 and the five-step procedure in Type 4 problems. In problem Types 1-3 the procedure is:

1. u -assignment
2. u -substitution
3. integration in terms of u
4. x -substitution

Example: $\int (x + 1)^4 dx$

1. u -assignment: $u = x + 1$
 $du = dx$

2. u -substitution: $\int u^4 du$

3. integration: $\frac{1}{5}u^5 + C$

4. x -substitution: $\frac{1}{5}(x + 1)^5 + C$

“Only he who never plays, never loses.”