The Weekly Rigor

No. 57 "A mathematician is a machine for turning coffee into theorems." July 25, 2015

51 Problems in Calculating Integrals Using *U*-Substitution with Solutions

(Part 8)

26. $$\int \sec^2\left(\tfrac{1}{3}x\right)dx = \int \sec^2(u)3du = 3\int \sec^2(u)du = 3\tan(u) + C =$$

$$= 3\tan\left(\tfrac{1}{3}x\right) + C.$$

$\left\{\begin{aligned} \text{Let } u &= \tfrac{1}{3}x \\ du &= \tfrac{1}{3}dx \\ 3du &= dx \end{aligned}\right.$

Check: $\left[3\tan\left(\tfrac{1}{3}x\right) + C\right]' = 3\sec^2\left(\tfrac{1}{3}x\right)\cdot\tfrac{1}{3} = \sec^2\left(\tfrac{1}{3}x\right)$. ✓

27. $$\int e^{2x+3}dx = \int e^u\frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x+3} + C.$$

$\left\{\begin{aligned} \text{Let } u &= 2x + 3 \\ du &= 2dx \end{aligned}\right.$

Check: $\left[\tfrac{1}{2}e^{2x+3} + C\right]' = \tfrac{1}{2}e^{2x+3}(2) = e^{2x+3}$. ✓

Type 3

28. $$\int (3x^2+1)^4 x\,dx = \int u^4\frac{du}{6} = \frac{1}{6}\int u^4 du = \frac{1}{6}\cdot\frac{1}{5}u^5 + C = \frac{1}{30}u^5 + C =$$

$$= \frac{1}{30}(3x^2+1)^5 + C.$$

$\left\{\begin{aligned} \text{Let } u &= 3x^2 + 1 \\ du &= 6xdx \\ \tfrac{du}{6} &= xdx \end{aligned}\right.$

Check: $\left[\tfrac{1}{30}(3x^2+1)^5 + C\right]' = \tfrac{5}{30}(3x^2+1)^4 6x = (3x^2+1)^4$. ✓

29. $$\int \left(\tfrac{1}{2}x^3 - 50\right)^6 x^2\,dx = \int u^6\frac{2}{3}du = \frac{2}{3}\int u^6 du = \frac{2}{3}\cdot\frac{1}{7}u^7 + C =$$

$$= \frac{2}{21}u^7 + C = \frac{2}{21}\left(\tfrac{1}{2}x^3 - 50\right)^7 + C.$$

$\left\{\begin{aligned} \text{Let } u &= \tfrac{1}{2}x^3 - 50 \\ du &= \tfrac{3}{2}x^2dx \\ \tfrac{2}{3}du &= x^2dx \end{aligned}\right.$

Check: $\left[\tfrac{2}{21}\left(\tfrac{1}{2}x^3 - 50\right)^7 + C\right]' = \tfrac{14}{21}\left(\tfrac{1}{2}x^3 - 50\right)^6\tfrac{3}{2}x^2 = \left(\tfrac{1}{2}x^3 - 50\right)^6 x^2$. ✓

30. $\int x^2 \sqrt[6]{\frac{1}{2}x^3-50}\,dx = \int \left(\frac{1}{2}x^3 - 50\right)^{\frac{1}{6}} x^2\,dx = \int u^{\frac{1}{6}} \frac{2}{3} du = \frac{2}{3}\int u^{\frac{1}{6}} du =$

Let $u = \frac{1}{2}x^3 - 50$
$du = \frac{3}{2}x^2 dx$
$\frac{2}{3}du = x^2 dx$

$= \frac{2}{3}\cdot\frac{6}{7}u^{\frac{7}{6}} + C = \frac{4}{7}u^{\frac{7}{6}} + C = \frac{4}{7}\left(\frac{1}{2}x^3 - 50\right)^{\frac{7}{6}} + C.$

Check: $\left[\frac{4}{7}\left(\frac{1}{2}x^3 - 50\right)^{\frac{7}{6}} + C\right]' = \frac{4}{7}\cdot\frac{7}{6}\left(\frac{1}{2}x^3 - 50\right)^{\frac{1}{6}}\frac{3}{2}x^2 = x^2\left(\frac{1}{2}x^3 - 50\right)^{\frac{1}{6}}$. ✓

31. $\int \frac{x}{(3x^2+2)^3}dx = \int \frac{1}{(3x^2+2)^3}x dx = \int \frac{1}{u^3}\frac{du}{6} = \frac{1}{6}\int \frac{1}{u^3}du = \frac{1}{6}\int u^{-3}du =$

Let $u = 3x^2 + 2$
$du = 6xdx$
$\frac{du}{6} = xdx$

$= \frac{1}{6}\cdot\frac{-1}{2}u^{-2} + C = \frac{-1}{12}u^{-2} + C = \frac{-1}{12}(3x^2 + 2)^{-2} + C.$

Check: $\left[\frac{-1}{12}(3x^2 + 2)^{-2} + C\right]' = \frac{1}{6}(3x^2 + 2)^{-3}6x = \frac{x}{(3x^2+2)^3}$. ✓

32. $\int \frac{x}{\sqrt{3x^2+2}}dx = \int \frac{x}{(3x^2+2)^{\frac{1}{2}}}dx = \int \frac{1}{(3x^2+2)^{\frac{1}{2}}}xdx = \int \frac{1}{u^{\frac{1}{2}}}\frac{du}{6} = \frac{1}{6}\int \frac{1}{u^{\frac{1}{2}}}du =$

Let $u = 3x^2 + 2$
$du = 6dx$
$\frac{du}{6} = dx$

$= \frac{1}{6}\int u^{-\frac{1}{2}}du = \frac{1}{6}\cdot\frac{2}{1}u^{\frac{1}{2}} + C = \frac{1}{3}u^{\frac{1}{2}} + C = \frac{1}{3}(3x^2 + 2)^{\frac{1}{2}} + C =$

$= \frac{1}{3}\sqrt{3x^2 + 2} + C.$

Check: $\left[\frac{1}{3}\sqrt{3x^2 + 2} + C\right]' = \left[\frac{1}{3}(3x^2 + 2)^{\frac{1}{2}} + C\right]' =$

$= \frac{1}{3}\cdot\frac{1}{2}(3x^2 + 2)^{-\frac{1}{2}}6x = \frac{1}{6}(3x^2 + 2)^{-\frac{1}{2}}6x = \frac{x}{(3x^2+2)^{\frac{1}{2}}} = \frac{x}{\sqrt{3x^2+2}}$. ✓

33. $\int \frac{x^2}{\sqrt[5]{\frac{3}{7}x^3-21}}dx = \int \frac{x^2}{\left(\frac{3}{7}x^3-21\right)^{\frac{1}{5}}}dx = \int \frac{1}{\left(\frac{3}{7}x^3-21\right)^{\frac{1}{5}}}x^2dx =$

Let $u = \frac{3}{7}x^3 - 21$
$du = \frac{9}{7}x^2dx$
$\frac{7}{9}du = x^2dx$

$= \int \frac{1}{u^{\frac{1}{5}}}\frac{7}{9}du = \frac{7}{9}\int \frac{1}{u^{\frac{1}{5}}}du = \frac{7}{9}\int u^{-\frac{1}{5}}du = \frac{7}{9}\cdot\frac{5}{4}u^{\frac{4}{5}} + C = \frac{35}{36}u^{\frac{4}{5}} + C =$

$= \frac{35}{36}\left(\frac{3}{7}x^3 - 21\right)^{\frac{4}{5}} + C.$

Check: $\left[\frac{35}{36}\left(\frac{3}{7}x^3 - 21\right)^{\frac{4}{5}} + C\right]' = \frac{35}{36}\cdot\frac{4}{5}\left(\frac{3}{7}x^3 - 21\right)^{-\frac{1}{5}}\frac{9}{7}x^2 = \frac{x^2}{\left(\frac{3}{7}x^3-21\right)^{\frac{1}{5}}} =$

$= \frac{x^2}{\sqrt[5]{\frac{3}{7}x^3-21}}$. ✓