The Weekly Rigor

No. 59 "A mathematician is a machine for turning coffee into theorems." August 8, 2015

51 Problems in Calculating Integrals Using *U*-Substitution with Solutions
(Part 10)

41. $\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$ $\left\{\begin{array}{l}\text{Let } u = \sin(x) \\ du = \cos(x)\,dx\end{array}\right.$

Check: $\left[\frac{1}{2}\sin^2(x) + C\right]' = \frac{2}{2}\sin(x)\cos(x) = \sin(x)\cos(x).$ ✓

Alternate Solution:

$\int \sin(x)\cos(x)\,dx = \int \cos(x)\sin(x)\,dx = \int u\frac{du}{-1} = -\int u\,du =$ $\left\{\begin{array}{l}\text{Let } u = \cos(x) \\ du = -\sin(x)\,dx \\ \frac{du}{-1} = \sin(x)\,dx\end{array}\right.$

$= -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2(x) + C.$

Check: $\left[-\frac{1}{2}\cos^2(x) + C\right]' = -\frac{2}{2}\cos(x)\left[-\sin(x)\right] = \sin(x)\cos(x).$ ✓

Remark: It may seem surprising that the two solutions give two different yet correct answers. Consider, however, that by the Pythagorean Identities

$$\frac{1}{2}\sin^2(x) + C = \frac{1}{2}\left[1 - \cos^2(x)\right] + C = \frac{1}{2} - \frac{1}{2}\cos^2(x) + C = -\frac{1}{2}\cos^2(x) + \left(C + \frac{1}{2}\right).$$

Hence, if we take the derivative of the expressions at the extreme ends of the latter equation, we get the same result, viz.,

$$\sin(x)\cos(x).$$

42. $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}dx = \int \frac{1}{\sin(x)}\cos(x)\,dx = \int \frac{1}{u}du =$ $\left\{\begin{array}{l}\text{Let } u = \sin(x) \\ du = \cos(x)\,dx\end{array}\right.$

$= \ln|u| + C = \ln|\sin(x)| + C.$

Check: $[\ln|\sin(x)| + C\,]' = \frac{\cos(x)}{\sin(x)} = \cot(x)$. ✓

Remark: A similar problem is calculating the integral of the tangent function:

$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx = \int \frac{1}{\cos(x)}\sin(x)\,dx = \int \frac{1}{u}\frac{du}{-1} =$ $\begin{array}{l}\text{Let } u = \cos(x) \\ du = -\sin(x)\,dx \\ \frac{du}{-1} = \sin(x)\,dx\end{array}$

$= -\int \frac{1}{u}du = -\ln|u| + C = -\ln|\cos(x)| + C = \ln\left|\cos^{-1}(x)\right| + C =$

$= \ln\left|\frac{1}{\cos(x)}\right| + C = \ln|\sec(x)| + C.$

Check: $[\ln|\sec(x)| + C\,]' = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan(x)$. ✓

Type 4

43. $\int (x+3)(x-1)^4\,dx = \int (u+4)u^4 du = \int (u^5+4u^4)\,du =$

$= \int u^5 du + \int 4u^4 du = \int u^5 du + 4\int u^4 du = \frac{1}{6}u^6 + \frac{4}{5}u^5 + C =$

$= \frac{1}{6}(x-1)^6 + \frac{4}{5}(x-1)^5 + C =$

Check: $\left[\frac{1}{6}(x-1)^6 + \frac{4}{5}(x-1)^5 + C\right]' = (x-1)^5 + 4(x-1)^4 =$

$= [(x-1)+4](x-1)^4 = (x+3)(x-1)^4$. ✓

$\left\{ \begin{array}{l} \text{Let } u = x-1 \\ du = dx \end{array} \right.$
$u + 4 = x + 3$

44. $\int x^5 \sqrt[5]{1+x^2}\,dx = \int x^4 \sqrt[5]{1+x^2} x dx = \int (u-1)^2 \sqrt[5]{u}\frac{du}{2} =$

$= \frac{1}{2}\int (u-1)^2 \sqrt[5]{u} du = \frac{1}{2}\int (u-1)^2 u^{\frac{1}{5}} du =$

$= \frac{1}{2}\int (u^2-2u+1)\, u^{\frac{1}{5}} du = \frac{1}{2}\int \left(u^{\frac{11}{5}} - 2u^{\frac{6}{5}} + u^{\frac{1}{5}}\right) du =$

$= \frac{1}{2}\int u^{\frac{11}{5}}\,du - \int u^{\frac{6}{5}}\,du + \frac{1}{2}\int u^{\frac{1}{5}}\,du =$

$= \frac{1}{2}\cdot\frac{5}{16}u^{\frac{16}{5}} - \frac{5}{11}u^{\frac{11}{5}} + \frac{1}{2}\cdot\frac{5}{6}u^{\frac{6}{5}} + C = \frac{5}{32}u^{\frac{16}{5}} - \frac{5}{11}u^{\frac{11}{5}} + \frac{5}{12}u^{\frac{6}{5}} + C =$

$= \frac{5}{32}(1+x^2)^{\frac{16}{5}} - \frac{5}{11}(1+x^2)^{\frac{11}{5}} + \frac{5}{12}(1+x^2)^{\frac{6}{5}} + C$.

Check: $\left[\frac{5}{32}(1+x^2)^{\frac{16}{5}} - \frac{5}{11}(1+x^2)^{\frac{11}{5}} + \frac{5}{12}(1+x^2)^{\frac{6}{5}} + C\ \right]' =$

$= \frac{5}{32}\cdot\frac{16}{5}(1+x^2)^{\frac{11}{5}}2x - \frac{5}{11}\cdot\frac{11}{5}(1+x^2)^{\frac{6}{5}}2x + \frac{5}{12}\cdot\frac{6}{5}(1+x^2)^{\frac{1}{5}}2x =$

$= (1+x^2)^{\frac{11}{5}}x - 2(1+x^2)^{\frac{6}{5}}x + (1+x^2)^{\frac{1}{5}}x =$

$= \left[(1+x^2)^{\frac{10}{5}} - 2(1+x^2)^{\frac{5}{5}} + 1\right](1+x^2)^{\frac{1}{5}}x =$

$= [(1+x^2)^2 - 2(1+x^2) + 1](1+x^2)^{\frac{1}{5}}x =$

$= (1+2x^2+x^4-2-2x^2+1)x\sqrt[5]{1+x^2} =$

$= x^5\sqrt[5]{1+x^2}$. ✓

$\left\{ \begin{array}{l} \text{Let } u = 1+x^2 \\ du = 2xdx \end{array} \right.$
$\frac{du}{2} = xdx$

$u - 1 = x^2$
$(u-1)^2 = (x^2)^2 = x^4$