The Weekly Rigor

No. 60 “A mathematician is a machine for turning coffee into theorems.” August 15, 2015

51 Problems in Calculating Integrals Using *U*-Substitution with Solutions
(Part 11)

45.
$$\int x\sqrt{x-1}\,dx = \int x(x-1)^{\frac{1}{2}}\,dx = \int (u+1)u^{\frac{1}{2}}\,du =$$
$$= \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = \int u^{\frac{3}{2}}\,du + \int u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C =$$
$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C.$$

Let $u = x - 1$, $du = dx$; $u + 1 = x$

Check: $\left[\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C\right]' = (x-1)^{\frac{3}{2}} + (x-1)^{\frac{1}{2}} =$
$$= \left[(x-1)^{\frac{2}{2}} + 1\right](x-1)^{\frac{1}{2}} = [(x-1)+1](x-1)^{\frac{1}{2}} = x(x-1)^{\frac{1}{2}} =$$
$$= x\sqrt{x-1}\,. \checkmark$$

46.
$$\int \frac{x}{\sqrt{1+2x}}\,dx = \int \frac{x}{(1+2x)^{\frac{1}{2}}}\,dx = \int \frac{\left(\frac{u-1}{2}\right)}{u^{\frac{1}{2}}}\frac{du}{2} = \frac{1}{2}\int \frac{u-1}{2u^{\frac{1}{2}}}\,du =$$
$$= \frac{1}{4}\int \frac{u-1}{u^{\frac{1}{2}}}\,du = \frac{1}{4}\int \frac{u}{u^{\frac{1}{2}}}\,du - \frac{1}{4}\int \frac{1}{u^{\frac{1}{2}}}\,du = \frac{1}{4}\int u^{\frac{1}{2}}\,du - \frac{1}{4}\int u^{\frac{-1}{2}}\,du =$$
$$= \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} - \frac{1}{4}\cdot\frac{2}{1}u^{\frac{1}{2}} + C = \frac{1}{6}u^{\frac{3}{2}} - \frac{1}{2}u^{\frac{1}{2}} + C =$$
$$= \frac{1}{6}(1+2x)^{\frac{3}{2}} - \frac{1}{2}(1+2x)^{\frac{1}{2}} + C.$$

Let $u = 1 + 2x$, $du = 2dx$; $\frac{du}{2} = dx$; $u - 1 = 2x$; $\frac{u-1}{2} = x$

Check: $\left[\frac{1}{6}(1+2x)^{\frac{3}{2}} - \frac{1}{2}(1+2x)^{\frac{1}{2}} + C\right]' =$
$$= \frac{1}{4}(1+2x)^{\frac{1}{2}}\cdot 2 - \frac{1}{4}(1+2x)^{\frac{-1}{2}}\cdot 2 = \frac{1}{2}(1+2x)^{\frac{1}{2}} - \frac{1}{2}(1+2x)^{\frac{-1}{2}} =$$
$$= \frac{(1+2x)^{\frac{1}{2}}}{2} - \frac{1}{2(1+2x)^{\frac{1}{2}}} = \frac{\sqrt{1+2x}}{2} - \frac{1}{2\sqrt{1+2x}} =$$
$$= \frac{1+2x}{2\sqrt{1+2x}} - \frac{1}{2\sqrt{1+2x}} = \frac{(1+2x)-1}{2\sqrt{1+2x}} = \frac{2x}{2\sqrt{1+2x}} = \frac{x}{\sqrt{1+2x}}\,. \checkmark$$

47. $\int \frac{x}{\sqrt[4]{x+2}}dx = \int \frac{1}{\sqrt[4]{x+2}} x dx = \int \frac{1}{\sqrt[4]{u}}(u-2)du =$

$= \int \left(\frac{u}{u^{\frac{1}{4}}} - \frac{2}{u^{\frac{1}{4}}}\right) du = \int \left(u^{\frac{3}{4}} - 2u^{\frac{-1}{4}}\right) du = \int u^{\frac{3}{4}} du - 2\int u^{\frac{-1}{4}} du =$

$= \frac{4}{7}u^{\frac{7}{4}} - 2\frac{4}{3}u^{\frac{3}{4}} + C = \frac{4}{7}u^{\frac{7}{4}} - \frac{8}{3}u^{\frac{3}{4}} + C = \frac{4}{7}(x+2)^{\frac{7}{4}} - \frac{8}{3}(x+2)^{\frac{3}{4}} + C$.

Let $u = x + 2$
$du = dx$
$u - 2 = x$

Check: $\left[\frac{4}{7}(x+2)^{\frac{7}{4}} - \frac{8}{3}(x+2)^{\frac{3}{4}} + C\right]' = (x+2)^{\frac{3}{4}} - 2(x+2)^{\frac{-1}{4}} =$

$= (x+2)^{\frac{3}{4}} - \frac{2}{(x+2)^{\frac{1}{4}}} = \frac{(x+2)-2}{(x+2)^{\frac{1}{4}}} = \frac{x}{\sqrt[4]{x+2}}$. ✓

48. $\int \frac{x+4}{2x+5}dx = \int \frac{1}{2x+5}(x+4)dx = \int \frac{1}{u}\cdot\frac{u+3}{2}\frac{du}{2} = \frac{1}{4}\int \frac{u+3}{u}du =$

$= \frac{1}{4}\int \left(\frac{u}{u} + \frac{3}{u}\right) du = \frac{1}{4}\int 1\, du + \frac{1}{4}\int \frac{3}{u} du = \frac{1}{4}\int 1\, du + \frac{3}{4}\int \frac{1}{u} du =$

$= \frac{1}{4}u + \frac{3}{4}\ln|u| + C = \frac{1}{4}(2x+5) + \frac{3}{4}\ln|2x+5| + C.$

Let $u = 2x + 5$
$du = 2dx$
$\frac{du}{2} = dx$
$u - 5 = 2x$
$\frac{u-5}{2} = x$
$\frac{u-5}{2} + 4 = x + 4$
$\frac{u+3}{2} = x + 4$

Check: $\left[\frac{1}{4}(2x+5) + \frac{3}{4}\ln|2x+5| + C\right]' =$

$= \left[\frac{1}{2}x + \frac{5}{4} + \frac{3}{4}\ln|2x+5| + C\right]' = \frac{1}{2} + \frac{3}{4}\cdot\frac{2}{2x+5} = \frac{1}{2} + \frac{3}{2(2x+5)} =$

$= \frac{2x+5}{2(2x+5)} + \frac{3}{2(2x+5)} = \frac{2x+8}{2(2x+5)} = \frac{2(x+4)}{2(2x+5)} = \frac{x+4}{2x+5}$. ✓

Alternate Solution:

$\int \frac{x+4}{2x+5}dx = \int \frac{u}{2u-3}du$. Using polynomial division,

Let $u = x + 4$
$du = dx$
$u - 4 = x$
$2u - 8 = 2x$
$2u - 3 = 2x + 5$

$$\begin{array}{r|l} & \frac{1}{2} + \frac{\frac{3}{2}}{2u-3} \\ 2u-3 & u + 0 \\ & u - \frac{3}{2} \\ \hline & \frac{3}{2} \end{array}$$

Hence,

$\int \frac{u}{2u-3} = \int \left(\frac{1}{2} + \frac{\frac{3}{2}}{2u-3}\right) du = \frac{1}{2}\int 1\, du + \frac{3}{2}\int \frac{1}{2u-3} du =$

$= \frac{1}{2}u + \frac{3}{2}\cdot\frac{1}{2}\ln|2u-3| + C = \frac{1}{2}u + \frac{3}{4}\ln|2u-3| + C =$

$= \frac{1}{2}(x+4) + \frac{3}{4}\ln|2(x+4)-3| + C = \frac{1}{2}(x+4) + \frac{3}{4}\ln|2x-5| + C.$

Check: $\left[\frac{1}{2}(x+4) + \frac{3}{4}\ln|2x-5| + C\right]' = \frac{1}{2} + \frac{3}{2}\cdot\frac{1}{2x+5} = \frac{2x+5+3}{2(2x+5)} =$

$= \frac{2x+8}{2(2x+5)} = \frac{2(x+4)}{2(2x+5)} = \frac{x+4}{2x+5}$. ✓