The Weekly Rigor

No. 61 "A mathematician is a machine for turning coffee into theorems." August 22, 2015

51 Problems in Calculating Integrals Using *U*-Substitution with Solutions
(Part 12)

49. $\int \frac{x^2+4}{x+2}dx = \int \frac{1}{x+2}(x^2+4)dx = \int \frac{1}{u}(u^2-4u+8)du =$

$= \int \left(u - 4 + \frac{8}{u}\right) du = \int u\, du - \int 4\, du + \int \frac{8}{u} du =$

$= \int u\, du - 4\int 1\, du + 8\int \frac{1}{u} du = \frac{1}{2}u^2 - 4u + 8\ln|u| + C =$

$= \frac{1}{2}(x+2)^2 - 4(x+2) + 8\ln|x+2| + C$.

Let $u = x+2$, $du = dx$

$u - 2 = x$

$(u-2)^2 = x^2$

$u^2 - 4u + 4 + 4 = x^2 + 4$

$u^2 - 4u + 8 = x^2 + 4$

Check: $\left[\frac{1}{2}(x+2)^2 - 4(x+2) + 8\ln|x+2| + C\right]' =$

$= x + 2 - 4 + \frac{8}{x+2} = x - 2 + \frac{8}{x+2} = \frac{(x-2)(x+2)+8}{x+2} =$

$= \frac{x^2-4+8}{x+2} = \frac{x^2+4}{x+2}$. ✓

50. $\int (x^3+1)^4 x^5\, dx = \int (x^3+1)^4 x^3 x^2\, dx = \int u^4(u-1)\frac{du}{3} =$

$= \frac{1}{3}\int u^4(u-1)\, du = \frac{1}{3}\int (u^5 - u^4)\, du = \frac{1}{3}\int u^5\, du - \frac{1}{3}\int u^4\, du =$

$= \frac{1}{3}\cdot\frac{1}{6}u^6 - \frac{1}{3}\cdot\frac{1}{5}u^5 + C = \frac{1}{18}u^6 - \frac{1}{15}u^5 + C =$

$= \frac{1}{18}(x^3+1)^6 - \frac{1}{15}(x^3+1)^5 + C.$

Let $u = x^3+1$, $du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$u - 1 = x^3$

Check: $\left[\frac{1}{18}(x^3+1)^6 - \frac{1}{15}(x^3+1)^5 + C\right]' =$

$= \frac{6}{18}(x^3+1)^5 3x^2 - \frac{5}{15}(x^3+1)^4 3x^2 = x^2(x^3+1)^5 - x^2(x^3+1)^4 =$

$= x^2(x^3+1)^4[x^3+1-1] = x^2(x^3+1)^4 x^3 = (x^3+1)^4 x^5.$ ✓

51. $\int \frac{(3+\ln(x))^2(2-\ln(x))}{x}dx = \int (3+\ln(x))^2(2-\ln(x))\frac{1}{x}dx =$

$= \int u^2(5-u)\,du = \int (5u^2-u^3)\,du = \int 5u^2\,du - \int u^3\,du =$

$= 5\int u^2\,du - \int u^3\,du = \frac{5}{3}u^3 - \frac{1}{4}u^4 + C =$

$= \frac{5}{3}(3+\ln(x))^3 - \frac{1}{4}(3+\ln(x))^4 + C.$

Let $u = 3+\ln(x)$
$du = \frac{1}{x}dx$
$u-3 = \ln(x)$
$3-u = -\ln(x)$
$5-u = 2-\ln(x)$

Check: $\left[\frac{5}{3}(3+\ln(x))^3 - \frac{1}{4}(3+\ln(x))^4 + C\right]' =$

$= 5(3+\ln(x))^2\left(\frac{1}{x}\right) - (3+\ln(x))^3\left(\frac{1}{x}\right) =$

$= \frac{5(3+\ln(x))^2 - (3+\ln(x))^3}{x} = \frac{(3+\ln(x))^2[5-(3+\ln(x))]}{x} =$

$= \frac{(3+\ln(x))^2(2-\ln(x))}{x}$. ✓

Alternate Solution:

$\int \frac{(3+\ln(x))^2(2-\ln(x))}{x}dx = \int (3+\ln(x))^2(2-\ln(x))\frac{1}{x}dx =$

$= \int (5-u)^2 u \frac{du}{-1} = -\int (25-10u+u^2)u\,du =$

$= -\int (25u-10u^2+u^3)\,du = \frac{-25}{2}u^2 + \frac{10}{3}u^3 - \frac{1}{4}u^4 + C =$

$= \frac{-25}{2}(2-\ln(x))^2 + \frac{10}{3}(2-\ln(x))^3 - \frac{1}{4}(2-\ln(x))^4 + C.$

Let $u = 2-\ln(x)$
$du = \frac{-1}{x}dx$
$\frac{du}{-1} = \frac{1}{x}dx$
$\ln(x) + u = 2$
$\ln(x) = 2-u$
$3+\ln(x) = 5-u$

Check: $\left[\frac{-25}{2}(2-\ln(x))^2 + \frac{10}{3}(2-\ln(x))^3 - \frac{1}{4}(2-\ln(x))^4 + C\right]' =$

$= -25(2-\ln(x))\left(\frac{-1}{x}\right) + 10(2-\ln(x))^2\left(\frac{-1}{x}\right) - (2-\ln(x))^3\left(\frac{-1}{x}\right) =$

$= \left(\frac{25}{x}\right)(2-\ln(x)) - \left(\frac{10}{x}\right)(2-\ln(x))^2 + \left(\frac{1}{x}\right)(2-\ln(x))^3 =$

$= \frac{1}{x}(2-\ln(x))[25-10(2-\ln(x))+(2-\ln(x))^2] =$

$= \frac{(2-\ln(x))}{x}\left[25-20+10\ln(x)+4-4\ln(x)+\ln^2(x)\right] =$

$= \frac{(2-\ln(x))}{x}\left[9+6\ln(x)+\ln^2(x)\right] = \frac{(2-\ln(x))(3+\ln(x))^2}{x} =$

$= \frac{(3+\ln(x))^2(2-\ln(x))}{x}$. ✓