The Weekly Rigor

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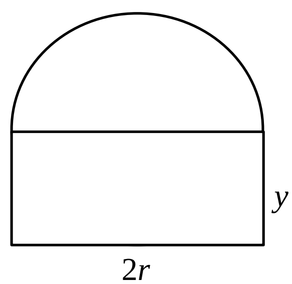
The Norman Window Problem

(Part 1)

Here the Norman window problem shall consist of the following: Find the maximum area of a Norman window with a given fixed perimeter. For example:

Problem: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 55 inches, what is the maximum area of the window?



Solution: The perimeter is the total distance around the outside of the window. Hence, we need to add up the following: the two sides of the rectangles, the bottom of the rectangle, and the arc length of the semicircle. So, since the perimeter was given as 55, we have the equation

$$55 = 2y + 2r + \pi r.$$

(Recall that the circumference of a whole circle is $2\pi r$. Later on we are going to use this equation to solve for y in terms of r.)

The window's area ("A") is the sum of the area of the rectangle and the semicircle. Hence, we have the equation

$$A = 2ry + \frac{1}{2}\pi r^2$$

(Recall that the area of a whole circle is πr^2.) To use this equation, we must replace the "y" with an expression in terms of r. That way we will have A as a function of r alone. Using the above equation of the perimeter,

$$55 = 2y + 2r + \pi r,$$

we derive

$$55 - 2r - \pi r = 2y,$$

i.e.,

$$\frac{55 - 2r - \pi r}{2} = y.$$

So, by substitution,

$$A(r) = 2r\left(\frac{55 - 2r - \pi r}{2}\right) + \tfrac{1}{2}\pi r^2 = r(55 - 2r - \pi r) + \tfrac{1}{2}\pi r^2 = 55r - 2r^2 - \pi r^2 + \tfrac{1}{2}\pi r^2.$$

Thus,

$$A(r) = -\tfrac{1}{2}\pi r^2 - 2r^2 + 55r = \left(-\tfrac{1}{2}\pi - 2\right)r^2 + 55r = -\left(\tfrac{1}{2}\pi + 2\right)r^2 + 55r =$$
$$= -\frac{(\pi + 4)}{2}r^2 + 55r.$$

Using the vertex formula,

$$r = \frac{-b}{2a} \text{ with } a = \frac{-(\pi+4)}{2} \text{ and } b = 55,$$

we have, letting "r_{max}" denote the radius that maximizes the area,

$$r_{max} = \frac{-55}{2\left(\frac{-(\pi+4)}{2}\right)} = \frac{55}{\pi+4}.$$

This radius will maximize the area of the window since it is located at the vertex of the quadratic area function. Observe that this radius equals the given perimeter divided by $\pi + 4$. Now consider the resulting maximal area at this value of the radius, where "A_{max}" denotes the maximal area:

$$A_{max} = A\left(\frac{55}{\pi+4}\right) = -\frac{(\pi + 4)}{2}\left(\frac{55}{\pi + 4}\right)^2 + 55\left(\frac{55}{\pi + 4}\right) = -\frac{1}{2}\frac{(55)^2}{(\pi + 4)} + \frac{(55)^2}{\pi + 4} = \frac{1}{2}\frac{(55)^2}{(\pi + 4)} =$$
$$= \frac{55}{2}\cdot\frac{55}{\pi + 4} = \frac{55}{2}\cdot r_{max}\,.$$

Note that this maximal area equals one half the given perimeter times the radius we found just above. These two observations suggest the generalization that for given perimeter P, $r_{max} = \frac{P}{\pi+4}$ and $A_{max} = \frac{P}{2}\cdot r_{max}$. We will prove this generalization in three ways: one using precalculus techniques, a second using methods taught in single-variable calculus, and a third approach using Lagrange multipliers, a subject investigated in multi-variable calculus.