# The Weekly Rigor

No. 68 "A mathematician is a machine for turning coffee into theorems." October 10, 2015


## The Norman Window Problem

(Part 2)

**Claim:** For a Norman window with the shape of a rectangle surmounted by a semicircle of radius $r$ and having perimeter $P$, the maximum area $A$ of the window is found at $r_{max} = \frac{P}{\pi+4}$ with value $A_{max} = \frac{P}{2} \cdot r_{max}$.

PROOF FROM PRECALCULUS

**Proof:** Let $y$ denote the height of the rectangle. Hence, $P = 2y + 2r + \pi r$. So, $y = \frac{P-2r-\pi r}{2}$. Furthermore, $A = 2ry + \frac{1}{2}\pi r^2$. Thus, by substitution, $A(r) =$
$= 2r\left(\frac{P-2r-\pi r}{2}\right) + \frac{1}{2}\pi r^2 = Pr - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = Pr - 2r^2 - \frac{1}{2}\pi r^2 =$
$= -\left(\frac{1}{2}\pi + 2\right)r^2 + Pr = -\frac{(\pi+4)}{2}r^2 + Pr$. Therefore, by the vertex formula $r_{max} = \frac{P}{\pi+4}$ and also $A_{max} = A(r_{max}) = A\left(\frac{P}{\pi+4}\right) = -\frac{(\pi+4)}{2}\left(\frac{P}{\pi+4}\right)^2 + P\left(\frac{P}{\pi+4}\right) = -\frac{1}{2}\frac{P^2}{(\pi+4)} + \frac{P^2}{\pi+4} = \frac{1}{2}\frac{P^2}{(\pi+4)} =$
$= \frac{P}{2} \cdot \frac{P}{\pi+4} = \frac{P}{2} \cdot r_{max}$, by substitution.

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PROOF FROM SINGLE-VARIABLE CALCULUS

**Proof:** Let $y$ denote the height of the rectangle. Hence, $P = 2y + 2r + \pi r$. So, $y = \frac{P-2r-\pi r}{2}$. Furthermore, $A = 2ry + \frac{1}{2}\pi r^2$. Thus, by substitution, $A(r) =$
$= 2r\left(\frac{P-2r-\pi r}{2}\right) + \frac{1}{2}\pi r^2 = Pr - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = Pr - 2r^2 - \frac{1}{2}\pi r^2 =$
$= -\left(\frac{1}{2}\pi + 2\right)r^2 + Pr = -\frac{(\pi+4)}{2}r^2 + Pr$. Hence, $A'(r) = -(\pi+4)r + P$. But $A'(r) = 0$ when $-(\pi+4)r + P = 0$, i.e., when $r = \frac{P}{\pi+4}$. Therefore, $r_{max} = \frac{P}{\pi+4}$ and also $A_{max} = A(r_{max}) =$
$= A\left(\frac{P}{\pi+4}\right) = -\frac{(\pi+4)}{2}\left(\frac{P}{\pi+4}\right)^2 + P\left(\frac{P}{\pi+4}\right) = -\frac{1}{2}\frac{P^2}{(\pi+4)} + \frac{P^2}{\pi+4} = \frac{1}{2}\frac{P^2}{(\pi+4)} = \frac{P}{2} \cdot \frac{P}{\pi+4} = \frac{P}{2} \cdot r_{max}$, by substitution.

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**Proof:** Let $y$ denote the height of the rectangle. Hence, $P = 2y + 2r + \pi r$. Furthermore, $A = 2ry + \frac{1}{2}\pi r^2$. So,

$$A_r = 2y + \pi r \qquad P_r = 2 + \pi$$
$$A_y = 2r \qquad P_y = 2$$

Thus,

$$2y + \pi r = \lambda(2 + \pi)$$
$$2r = 2\lambda$$
$$P = 2y + 2r + \pi r$$

Hence, $\lambda = \frac{2y+\pi r}{\pi+2}$ and $\lambda = r$. So, $\frac{2y+\pi r}{\pi+2} = r \implies 2y + \pi r = r(\pi + 2) \implies$ $\implies 2y + \pi r = r\pi + 2r \implies y = r$. Consequently, $P = 2r + 2r + \pi r = r(\pi + 4)$, by substitution. Therefore, $r_{max} = \frac{P}{\pi+4}$ and also $A_{max} = A(r_{max}) = A\left(\frac{P}{\pi+4}\right) =$ $= -\frac{(\pi+4)}{2}\left(\frac{P}{\pi+4}\right)^2 + P\left(\frac{P}{\pi+4}\right) = -\frac{1}{2}\frac{P^2}{(\pi+4)} + \frac{P^2}{\pi+4} = \frac{1}{2}\frac{P^2}{(\pi+4)} = \frac{P}{2}\cdot\frac{P}{\pi+4} = \frac{P}{2}\cdot r_{max}$, by substitution.

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