The Weekly Rigor

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The Two-Sided Limit Test for Single-Variable Calculus

Definition 1: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the *limit of f(x) as x approaches a is* L, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Definition 2:

$$\lim_{x\to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } 0 < x - a < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Definition 3:

$$\lim_{x\to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } 0 < a - x < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Theorem 1: If $\lim_{x\to a} f(x) = L$ then $\lim_{x\to a^+} f(x) = L$.

Proof: Suppose that $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. Hence, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon,$$

by Definition 1. Suppose that $0 < x - a < \delta$. So, $0 < |x - a| < \delta$. Thus, $|f(x) - L| < \varepsilon$. Consequently, if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$. Therefore, $\lim_{x\to a^+} f(x) = L$, by Definition 2.

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Theorem 2: If $\lim_{x\to a} f(x) = L$ then $\lim_{x\to a^-} f(x) = L$.

Proof: Suppose that $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. Hence, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon,$$

by Definition 1. Suppose that $0 < a - x < \delta$. So, $0 < |x - a| < \delta$. Thus, $|f(x) - L| < \varepsilon$. Consequently, if $0 < a - x < \delta$, then $|f(x) - L| < \varepsilon$. Therefore, $\lim_{x\to a^-} f(x) = L$, by Definition 3.

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Theorem 3: If $\lim_{x\to a} f(x) = L$ then $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

Proof: By Theorems 2 and 3.

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Theorem 4: If $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$, then $\lim_{x\to a} f(x) = L$.

Proof: Suppose that $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$. Let $\varepsilon > 0$. Hence, there exists $\delta_1 > 0$ such that if $0 < x - a < \delta_1$, then $|f(x) - L| < \varepsilon$, by Definition 2. Furthermore, there exists $\delta_2 > 0$ such that if $0 < a - x < \delta_2$, then $|f(x) - L| < \varepsilon$, by Definition 3. Let $\delta = min\{\delta_1, \delta_2\}$. Suppose that $0 < |x - a| < \delta$. So, either $0 < x - a < \delta_1$ or $0 < a - x < \delta_2$.

Case 1: Suppose that $0 < x - a < \delta_1$. Hence, $|f(x) - L| < \varepsilon$.

Case 2: Suppose that $0 < a - x < \delta_2$. Hence, $|f(x) - L| < \varepsilon$.

In either case, $|f(x) - L| < \varepsilon$. Consequently, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Therefore, $\lim_{x\to a} f(x) = L$, by Definition 1.

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Theorem 5 (The Two-Sided Limit Test)**:** $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L =$
$= \lim_{x\to a^-} f(x)$.

Proof: By Theorems 3 and 4.

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Examples: a.) Let $f(x) = |x|$. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$. Furthermore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = -0 = 0$. Hence, $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x)$. Therefore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0$, by the Two-Sided Limit Test.

b.) Let $f(x) = \frac{|x|}{x}$. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$. Furthermore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$. Hence, $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$. Therefore,

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{|x|}{x}$, does not exist, by the Two-Sided Limit Test.