The Weekly Rigor

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Limits of Real Number Sequences and Their Reciprocals
(Part 1)

Definition 1: A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N, \text{ then } |a_n - L| < \varepsilon.$$

Definition 2: $\lim_{n\to\infty} a_n = +\infty$ means that for every $M > 0$ there is an integer N such that

$$\text{if } n > N, \text{ then } a_n > M.$$

Definition 3: $\lim_{n\to\infty} a_n = -\infty$ means that for every $M < 0$ there is an integer N such that

$$\text{if } n > N, \text{ then } a_n < M.$$

Theorem 1: $\lim_{n\to\infty} a_n = L \neq 0$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$.

Proof: Suppose that $\lim_{n\to\infty} a_n = L \neq 0$. $\lim_{n\to\infty} 1 = 1$. Therefore, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} a_n} = \frac{1}{L}$, by the limit laws and substitution.

Suppose that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$. Hence, $L \neq 0$ and $a_n \neq 0$ for any n. So, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} a_n} = = \frac{1}{\lim_{n\to\infty} a_n}$, by the limit laws. Consequently, $\frac{1}{\lim_{n\to\infty} a_n} = \frac{1}{L}$. Therefore, $\lim_{n\to\infty} a_n = L \neq 0$.

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Remark: In the following theorems, each term of the sequence a_n is a positive real number.

Theorem 2: $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$.

Proof: Suppose that $\lim_{n\to\infty} a_n = 0$. Let $M > 0$ be given and set $\varepsilon = \frac{1}{M}$. Hence, $\varepsilon > 0$. So, there exists an integer *N* such that if $n > N$, then $|a_n - 0| < \varepsilon = \frac{1}{M}$, by Definition 1. Thus, since $a_n > 0$, if $n > N$, then $0 < a_n < \frac{1}{M}$. Hence, if $n > N$, then $\frac{1}{a_n} > M$. Consequently, $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$, by Definition 2. Therefore, if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$.

Suppose that $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$. Let $\varepsilon > 0$ be given and set $M = \frac{1}{\varepsilon}$. Hence, $M > 0$. So, there exists an integer *N* such that if $n > N$, then $\frac{1}{a_n} > M = \frac{1}{\varepsilon}$, by Definition 2. Thus, since $a_n > 0$, if $n > N$, then $a_n < \varepsilon$. Hence, if $n > N$, then $|a_n - 0| < \varepsilon$. Consequently, $\lim_{n\to\infty} a_n = 0$, by Definition 1. Therefore, if $\lim_{n\to\infty} \frac{1}{a_n} = +\infty$, then $\lim_{n\to\infty} a_n = 0$.

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Theorem 3: $\lim_{n\to\infty} -a_n = 0$ if and only if $\lim_{n\to\infty} \frac{-1}{a_n} = -\infty$.

Proof: Suppose that $\lim_{n\to\infty} -a_n = 0$. Let $M < 0$ be given and set $\varepsilon = \frac{-1}{M}$. Hence, $\varepsilon > 0$. So, there exists an integer *N* such that if $n > N$, then $|-a_n - 0| < \varepsilon = \frac{-1}{M}$, by Definition 1. Thus, since $a_n > 0$, if $n > N$, then $a_n < \varepsilon = \frac{-1}{M}$. Hence, if $n > N$, then $\frac{-1}{a_n} < M$. Consequently, $\lim_{n\to\infty} \frac{-1}{a_n} = -\infty$, by Definition 3. Therefore, if $\lim_{n\to\infty} - a_n = 0$, then $\lim_{n\to\infty} \frac{-1}{a_n} = -\infty$.

Suppose that $\lim_{n\to\infty} \frac{-1}{a_n} = -\infty$. Let $\varepsilon > 0$ be given and set $M = \frac{-1}{\varepsilon}$. Hence, $M < 0$. So, there exists an integer *N* such that if $n > N$, then $\frac{-1}{a_n} < M = \frac{-1}{\varepsilon}$, by Definition 3. Thus, since $a_n > 0$, if $n > N$, then $-\varepsilon < -a_n < 0$. Hence, if $n > N$, then $0 < a_n < \varepsilon$. But $a_n = |a_n| =$ $= |-a_n|$. So, if $n > N$, then $|-a_n - 0| < \varepsilon$, by substitution. Consequently, $\lim_{n\to\infty} -a_n = 0$, by Definition 1. Therefore, if $\lim_{n\to\infty} \frac{-1}{a_n} = -\infty$, then $\lim_{n\to\infty} -a_n = 0$.

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