The Weekly Rigor

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Limits of Real Number Sequences and Their Reciprocals

(Part 2)

Theorem 4: $\lim_{n\to\infty} \frac{1}{a_n} = 0$ if and only if $\lim_{n\to\infty} a_n = +\infty$.

Proof: Suppose that $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Let $M > 0$ be given and set $\varepsilon = \frac{1}{M}$. Hence, $\varepsilon > 0$. So, there exists an integer N such that if $n > N$, then $\left|\frac{1}{a_n} - 0\right| < \varepsilon = \frac{1}{M}$, by Definition 1. Thus, since $a_n > 0$, if $n > N$, then $0 < \frac{1}{a_n} < \frac{1}{M}$. Hence, if $n > N$, then $a_n > M$. Consequently, $\lim_{n\to\infty} a_n = +\infty$, by Definition 2. Therefore, if $\lim_{n\to\infty} \frac{1}{a_n} = 0$, then $\lim_{n\to\infty} a_n = +\infty$.

Suppose that $\lim_{n\to\infty} a_n = +\infty$. Let $\varepsilon > 0$ be given and set $M = \frac{1}{\varepsilon}$. Hence, $M > 0$. So, there exists an integer N such that if $n > N$, then $a_n > M = \frac{1}{\varepsilon}$, by Definition 2. Thus, since $a_n > 0$, if $n > N$, then $\varepsilon > \frac{1}{a_n} > 0$. Hence, if $n > N$, then $\left|\frac{1}{a_n} - 0\right| < \varepsilon$. Consequently, $\lim_{n\to\infty} \frac{1}{a_n} = 0$, by Definition 1. Therefore, if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

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Theorem 5: $\lim_{n\to\infty} \frac{-1}{a_n} = 0$ if and only if $\lim_{n\to\infty} - a_n = -\infty$.

Proof: Suppose that $\lim_{n\to\infty} \frac{-1}{a_n} = 0$. Let $M < 0$ be given and set $\varepsilon = \frac{-1}{M}$. Hence, $\varepsilon > 0$. So, there exists an integer N such that if $n > N$, then $\left|\frac{-1}{a_n} - 0\right| < \varepsilon = \frac{-1}{M}$, by Definition 1. But $\left|\frac{-1}{a_n} - 0\right| = \left|\frac{-1}{a_n}\right| = \frac{1}{a_n}$. Thus, since $a_n > 0$, if $n > N$, then $\frac{1}{a_n} < \varepsilon = \frac{-1}{M}$, by substitution. Hence, if $n > N$, then $a_n > -M$. So, , if $n > N$, then $-a_n < M$. Consequently, $\lim_{n\to\infty} -a_n = -\infty$, by Definition 3. Therefore, if $\lim_{n\to\infty} \frac{-1}{a_n} = 0$, then $\lim_{n\to\infty} -a_n = -\infty$.

Suppose that $\lim_{n\to\infty} -a_n = -\infty$. Let $\varepsilon > 0$ be given and set $M = \frac{-1}{\varepsilon}$. Hence, $M < 0$. So, there exists an integer N such that if $n > N$, then $-a_n < M = \frac{-1}{\varepsilon}$, by Definition 3. Thus, since $a_n > 0$, if $n > N$, then $-\varepsilon < \frac{-1}{a_n} < 0$. Hence, if $n > N$, then $0 < \frac{1}{a_n} < \varepsilon$. But $\left|\frac{-1}{a_n}\right| = \frac{1}{a_n}$. So, if $n > N$, then $\left|\frac{-1}{a_n} - 0\right| < \varepsilon$, by substitution. Consequently, $\lim_{n\to\infty} \frac{-1}{a_n} = 0$, by Definition 1. Therefore, if $\lim_{n\to\infty} -a_n = -\infty$, then $\lim_{n\to\infty} \frac{-1}{a_n} = 0$.

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