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Pedagogical Introduction to the Fundamental Horizontal Asymptotes of Rational Functions

(Part 1)

INTRODUCTION

A rational function is a function consisting of the ratio of two polynomials. For example, $f(x) = \frac{2x^3+4x^2+3x+1}{5x^3+x^2+6x}$. For rational functions, a horizontal asymptote is a *y*-value the function gets ever closer to as *x* grows arbitrarily large in either the positive or negative direction. (We will restrict our attention to *x* going in the positive direction.) In the case of the aforementioned function, it so happens that the *y*-values of *f*(*x*) get closer to $\frac{2}{5}$ as *x* grows arbitrarily large ("goes to positive infinity"). Algebraic methods of determining whether a rational function is asymptotic--and if so to which *y*-value--depend upon the principle that a ratio consisting of a fixed number divided by an ever increasing value of *x* is a ratio that has 0 as its horizontal asymptote. For illustration, consider the *y*-values of $\frac{1}{x}$ as *x* takes on the values of 10, 100, 1000, etc.: As *x* increases, *y* decreases closer and closer to 0. In the formal language of contemporary mathematics, $\lim_{x\to+\infty} \frac{1}{x} = 0$.

The purpose of this and the following article is to formally prove the basic asymptotic principles underlying the algebraic solution of rational functions. In this article the proofs will be more concrete whereas the next article will generalize the principles. Both articles will employ the standard formal limit definition of horizontal asymptotes.

Definition 1: Let *f* be a function that is defined on some infinite open interval $(a, +\infty)$. We shall write

$$\lim_{x\to+\infty} f(x) = L$$

if given any number $\varepsilon > 0$, there corresponds a positive number *N* such that

if $x > N$, then $|f(x) - L| < \varepsilon$.

To clarify Definition 1, consider again the function $f(x) = \frac{1}{x}$. In Theorem 1 we will see that $y = 0$ is a horizontal asymptote for this function, viz., $\lim_{x\to+\infty} \frac{1}{x} = 0$. Hence, we can use this fact to insert particular numbers into Definition 1 as applied to $f(x) = \frac{1}{x}$. Since the definition holds for any positive number ε, let us start with $\varepsilon = \frac{1}{2}$. Note that $L = 0$. The definition promises that there exists a positive number N such that all values of x greater than N necessitate that the distance between $f(x)$ and L—which in this case means $\frac{1}{x}$ and 0—is less than $\varepsilon = \frac{1}{2}$. In other words, the definition promises that there exists a positive number N such that

$$\text{if } x > N, \text{ then } \left|\frac{1}{x} - 0\right| < \frac{1}{2}.$$

Since $x > 0$, this conditional statement can be simplified to

$$\text{if } x > N, \text{ then } \frac{1}{x} < \frac{1}{2}.$$

A choice of N now readily suggests itself—in fact a whole range of choices—for any N at least as large as 2 will do. For simplicity, let $N = 2$. Thus, we have

$$\text{if } x > 2, \text{ then } \frac{1}{x} < \frac{1}{2},$$

from which we can say that $f(x)$ is bounded above by $\frac{1}{2}$ for all x-values greater than 2.

Furthermore, we can choose a smaller ε, say, $\varepsilon = \frac{1}{10}$ and by letting $N = 10$ construct a similar inequality,

$$\text{if } x > 10, \text{ then } \frac{1}{x} < \frac{1}{10},$$

which indicates that $f(x)$ is bounded above by $\frac{1}{10}$ for all x-values greater than 10. Finally, by choosing a tiny $\varepsilon = \frac{1}{1000}$ and allowing for a larger $N = 1000$ we have

$$\text{if } x > 1000, \text{ then } \frac{1}{x} < \frac{1}{1000},$$

resulting in an upper bound of $\frac{1}{1000}$ for $f(x)$.

We generalize these observations with the following theorem.

Theorem 1:

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Proof: Let $\varepsilon > 0$ be given and set $N = \frac{1}{\varepsilon}$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{1}{\varepsilon}$, by substitution. So, $\frac{1}{x} < \varepsilon$. Thus, since $x > 0$, $\left|\frac{1}{x}\right| < \varepsilon$. Hence, $\left|\frac{1}{x} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{1}{x} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \frac{1}{x} = 0$, by Definition 1.

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