The Weekly Rigor

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Pedagogical Introduction to the Fundamental Horizontal Asymptotes of Rational Functions

(Part 2)

Having proven that $\lim_{x \to +\infty} \frac{1}{x} = 0$, we may wonder what effect increasing the power of the x in the denominator may have on the value of the horizontal asymptote. The next theorem begins to address that question. Note that the proof depends on the relationship of $x^2 > x$, which itself requires that x be greater than 1. Hence the assignment of N to be at least as great as 1.

Theorem 2: $$\lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

Proof: Let $\varepsilon > 0$ be given and set $N = max\left\{1, \frac{1}{\varepsilon}\right\}$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{1}{\varepsilon}$, by substitution. Furthermore, $x > 1$. So, $x^2 > x$. Thus, $x^2 > \frac{1}{\varepsilon}$. Hence, $\frac{1}{x^2} < \varepsilon$. So, $\left|\frac{1}{x^2}\right| < \varepsilon$. Thus, $\left|\frac{1}{x^2} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{1}{x^2} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \frac{1}{x^2} = 0$, by Definition 1.

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Next we may wonder what effect a larger positive constant in the numerator will have on the value of the horizontal asymptote. That is the purpose of the next theorem.

Theorem 3: $$\lim_{x \to +\infty} \frac{4}{x} = 0.$$

Proof: Let $\varepsilon > 0$ be given and set $N = \frac{4}{\varepsilon}$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{4}{\varepsilon}$, by substitution. So, $\frac{1}{x} < \frac{\varepsilon}{4}$. Thus, $\frac{4}{x} < \varepsilon$. Thus, since $x > 0$, $\left|\frac{4}{x}\right| < \varepsilon$. Hence, $\left|\frac{4}{x} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{4}{x} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \frac{4}{x} = 0$, by Definition 1.

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Next let us put together the augmentation both of the power of the x in the denominator with the value of the positive constant in the numerator to see if any change in the horizontal asymptote results.

Theorem 4: $$\lim_{x\to+\infty}\frac{4}{x^2}=0.$$

Proof: Let $\varepsilon>0$ be given and set $N=max\left\{1,\frac{4}{\varepsilon}\right\}$. Hence, $N>0$.

Suppose that $x>N$. Hence, $x>\frac{4}{\varepsilon}$, by substitution. Furthermore, $x>1$. So, $x^2>x$. Thus, $x^2>\frac{4}{\varepsilon}$. Hence, $\frac{1}{x^2}<\frac{\varepsilon}{4}$. So, $\frac{4}{x^2}<\varepsilon$. Thus, $\left|\frac{4}{x^2}\right|<\varepsilon$. Hence, $\left|\frac{4}{x^2}-0\right|<\varepsilon$. Consequently, if $x>N$, then $\left|\frac{4}{x^2}-0\right|<\varepsilon$.

Therefore, $\lim_{x\to+\infty}\frac{4}{x^2}=0$, by Definition 1.

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Finally, we may wonder if there is any significant difference when the constant in the numerator is a negative number.

Theorem 5: $$\lim_{x\to+\infty}\frac{-4}{x^2}=0.$$

Proof: Let $\varepsilon>0$ be given and set $N=max\left\{1,\frac{4}{\varepsilon}\right\}$. Hence, $N>0$.

Suppose that $x>N$. Hence, $x>\frac{4}{\varepsilon}$, by substitution. Furthermore, $x>1$. So, $x^2>x$. Thus, $x^2>\frac{4}{\varepsilon}$. Hence, $\frac{1}{x^2}<\frac{\varepsilon}{4}$. So, $\frac{4}{x^2}<\varepsilon$. But $\left|\frac{-4}{x^2}\right|=\frac{4}{x^2}$. Thus, $\left|\frac{-4}{x^2}\right|<\varepsilon$. Hence, $\left|\frac{-4}{x^2}-0\right|<\varepsilon$. Consequently, if $x>N$, then $\left|\frac{-4}{x^2}-0\right|<\varepsilon$.

Therefore, $\lim_{x\to+\infty}\frac{-4}{x^2}=0$, by Definition 1.

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