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Introduction to the Fundamental Horizontal Asymptotes of Rational Functions

INTRODUCTION

This article carries on in the spirit of the previous article by generalizing the principles proved in the latter.

Definition 1: Let *f* be a function that is defined on some infinite open interval $(a, +\infty)$. We shall write

$$\lim_{x\to+\infty} f(x) = L$$

if given any number $\varepsilon > 0$, there corresponds a positive number *N* such that

$$\text{if } x > N, \text{ then } |f(x) - L| < \varepsilon.$$

Theorem 1: $\lim_{x\to+\infty} \frac{c}{x^n} = 0$ for $c \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

Proof: Let $\varepsilon > 0$ be given and set $N = max\left\{1, \frac{c}{\varepsilon}\right\}$, where $c \in \mathbb{R}^+$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{c}{\varepsilon}$, by substitution. Furthermore, $x > 1$. So, $x^n > x$ for $n \in \mathbb{Z}^+$. Thus, $x^n > \frac{c}{\varepsilon}$. Hence, $\frac{1}{x^n} < \frac{\varepsilon}{c}$. So, $\frac{c}{x^n} < \varepsilon$. Thus, $\left|\frac{c}{x^n}\right| < \varepsilon$. Hence, $\left|\frac{c}{x^n} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{c}{x^n} - 0\right| < \varepsilon$.

Therefore, $\lim_{x\to+\infty} \frac{c}{x^n} = 0$ for $c \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, by Definition 1.

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Remark: Cf. Theorem 4 of *WR* no. 73.

Theorem 2: $\lim_{x \to +\infty} \frac{-c}{x^n} = 0$ for $c \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

Proof: Let $\varepsilon > 0$ be given and set $N = max\left\{1, \frac{c}{\varepsilon}\right\}$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{c}{\varepsilon}$, by substitution. Furthermore, $x > 1$. So, $x^n > x$ for $n \in \mathbb{Z}^+$. Thus, $x^n > \frac{c}{\varepsilon}$. Hence, $\frac{1}{x^n} < \frac{\varepsilon}{c}$. So, $\frac{c}{x^n} < \varepsilon$. But $\left|\frac{-c}{x^n}\right| = \frac{c}{x^n}$. Thus, $\left|\frac{-c}{x^n}\right| < \varepsilon$. Hence, $\left|\frac{-c}{x^n} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{-c}{x^n} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \frac{-c}{x^n} = 0$, by Definition 1.

■

Remark: Cf. Theorem 5 of *WR* no. 73.

Theorem 3: $\lim_{x \to +\infty} \frac{0}{x^n} = 0$ for $n \in \mathbb{Z}^+$.

Proof: Let $\varepsilon > 0$ be given and set $N = \frac{1}{\varepsilon}$. Hence, $N > 0$.

Suppose that $x > N$. Hence, $x > \frac{1}{\varepsilon}$, by substitution. So, $\frac{1}{x} < \varepsilon$. Thus, since $x > 0$, $\left|\frac{0}{x}\right| = \frac{0}{x} < \frac{1}{x} < \varepsilon$. Hence, $\left|\frac{0}{x} - 0\right| < \varepsilon$. Consequently, if $x > N$, then $\left|\frac{0}{x} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \frac{0}{x} = 0$, by Definition 1.

■

Theorem 4: $\lim_{x \to +\infty} \frac{c}{x^n} = 0$ for $c \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Proof: By Theorems 1-3.

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